A DIRICHLET PRINCIPLE FOR NON REVERSIBLE MARKOV CHAINS AND SOME RECURRENCE THEOREMS

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Abstract. We extend the Dirichlet principle to non-reversible Markov processes on countable state spaces. We present two variational formulas for the solution of the Poisson equation or, equivalently, for the capacity between two disjoint sets. As an application we prove a some recurrence theorems. In particular, we show the recurrence of two-dimensional cycle random walks under a second moment condition on the winding numbers.

1. Introduction

Since Kakutani work [10], probability theory has not only proven to be a powerful tool inside potential theory, but potential theory has also given deep insight into the study of Markov processes. For example, the Dirichlet and the Thomson principles, which express escape probabilities as infima and suprema, respectively, give efficient recurrence and transience criteria. One can use the Dirichlet principle to prove the recurrence of random walks in random conductances in dimension one and two, and the Thomson principle to prove transience in dimension larger than or equal to three [15, 17].

Recently, potential theory and the Dirichlet principle played an important role in the proof of almost sure convergence of Dirichlet functions in transient Markov processes [1, 6], and in the proof of the recurrence of a simple random walk on the trace of transient Markov processes [5]. In a completely different context, the Dirichlet principle has been a basic tool in the investigation of metastability of reversible Markov processes (cf. [4, 3] and references therein).

Most applications of potential theory to Markov processes, as the ones cited above, are however restricted to *reversible* processes due to the lack of variational formulas for the effective resistance between two sets in non-reversible processes. We fill this gap here, presenting a Dirichlet principle for general Markov processes in discrete state spaces.

To illustrate the interest of the Dirichlet principle, we present some direct implications of this result. In Lemma 2.8, we extend to non-reversible transient Markov processes a well known pointwise estimate of a function in terms of its Dirichlet form and the Green function. In the last section, we state some recurrence theorems for non-reversible processes. In particular, we show that the recurrence property of Durrett multidimensional generalization [8] of Sinai random walk relies in fact on the scale invariance properties of a stationary measure and not on the reversibility of the process. We also give a sufficient second moment condition for the recurrence of two-dimensional cyclic random walks considered before in [11, 13, 7, 12].

Key words and phrases. Markov process, Potential theory, Non-reversible, Dirichlet principle, Recurrence.

In a completely different direction, relying on the Dirichlet principle presented in this article we prove in [9] the metastable behavior of the condensate in supercritical asymmetric zero range processes, extending to the non-reversible case the results proved in [3].

2. Notation and Main results

Consider an irreducible Markov process $\{X_t : t \ge 0\}$ on a countable state space E with generator L. Denote by $\lambda(x)$, $x \in E$, the holding rates, by $p(x,y)$, $x \neq y \in E$, the jump probabilities, and by $r(x,y) = \lambda(x)p(x,y)$ the jump rates. In particular, for every function $f : E \to \mathbb{R}$ with finite support,

$$(Lf)(x) \;=\; \sum_{y\in E} r(x,y)[f(y) - f(x)] \;, \quad x \in E \;. \tag{2.1}$$

Note that Lf is well defined for a bounded function f.

Assume that the Markov process $\{X_t : t \ge 0\}$ admits a stationary state μ. Let $L^2(\mu)$ be the space of square summable functions $f : E \to \mathbb{R}$ endowed with the scalar product defined by

$$\langle f, g\rangle_\mu \;=\; \sum_{x\in E} \mu(x)\, f(x)\, g(x) \;.$$

Denote by the same symbol L the generator acting on a domain of $L^2(\mu)$, and by $D(f)$ the Dirichlet form or energy of a function $f : E \to \mathbb{R}$:

$$D(f) \;=\; \frac{1}{2} \sum_{x,y\in E} \mu(x)\, r(x,y)\, [f(y) - f(x)]^2$$

so that for f in the domain of the generator we have $D(f) = \langle f, (-L)f\rangle_\mu$.

For each $x \in E$, denote by $\mathbb{P}_x$ the probability measure on the path space $D(\mathbb{R}_+, E)$ of right continuous trajectories with left limits induced by the Markov process X_t starting from x. Expectation with respect to $\mathbb{P}_x$ is denoted by $\mathbb{E}_x$.

Denote by $\{X^*_t : t \ge 0\}$ the stationary Markov process X_t reversed in time. We shall refer to X^*_t as the adjoint or the time reversed process. It is well known that X^*_t is a Markov process on E whose generator L^* is the adjoint of L in $L^2(\mu)$. The jump rates $r^*(x,y)$, $x \neq y \in E$, of the adjoint process satisfy the balanced equations

$$\mu(x)\, r(x,y) \;=\; \mu(y)\, r^*(y,x) \;. \tag{2.2}$$

Denote by $\lambda^*(x) = \lambda(x)$, $x \in E$, $p^*(x,y)$, $x \neq y \in E$, the holding rates and the jump probabilities of the time reversed process X^*_t.

As above, for each $x \in E$, denote by $\mathbb{P}^*_x$ the probability measure on the path space $D(\mathbb{R}_+, E)$ induced by the Markov process X^*_t starting from x. Expectation with respect to $\mathbb{P}^*_x$ is denoted by $\mathbb{E}^*_x$.

For a subset A of E, denote by T_A (resp. T^+_A) the hitting (resp. return) time of a set A:

$$T_A \;:=\; \inf\big\{ s > 0 : X_s \in A \big\} \;,$$
$$T^+_A \;:=\; \inf\{ t > 0 : X_t \in A, X_s \neq X_0 \;\text{ for some } 0 < s < t\} \;.$$

When the set A is a singleton $\{a\}$, we denote $T_{\{a\}}$, $T^+_{\{a\}}$ by T_a, T^+_a, respectively. We set for every x in E, $M(x) = \mu(x)\lambda(x)$.

Definition 2.1. *For two disjoint subsets A, B of E, the capacity between A and B is defined as*

$$\mathrm{cap}(A,B) \;=\; \sum_{x\in A} M(x)\, \mathbb{P}_x\big[T_A^+ > T_B^+\big]\;. \tag{2.3}$$

Clearly, the sum may be infinite. We prove below in Lemma 2.3 that the capacity is symmetric: $\mathrm{cap}(A,B) = \mathrm{cap}(B,A)$.

We may also express the capacity in terms of the distribution of the adjoint process. By (2.2), for any sequence $x_0, x_1, \dots, x_n$ such that $p(x_i, x_{i+1}) > 0$, $0 \le i < n$, $M(x_0)\prod_{0\le i<n} p(x_i,x_{i+1}) = M(x_n)\prod_{0\le i<n} p^*(x_{i+1},x_i)$. It follows from this observation that for any $a\in A$, $b\in B$, $M(a)\mathbb{P}_a[T_B < T_A^+, T_B = T_b] = M(b)\mathbb{P}^*_b[T_A < T_B^+, T_A = T_a]$. Hence,

$$\mathrm{cap}(A,B) \;=\; \sum_{a\in A} M(a)\mathbb{P}_a[T_B < T_A^+] \;=\; \sum_{a\in A}\sum_{b\in B} M(a)\mathbb{P}_a[T_B < T_A^+, T_B = T_b]\;,$$

so that

$$\mathrm{cap}(A,B) \;=\; \sum_{b\in B} M(b)\, \mathbb{P}^*_b[T_A^+ < T_B^+] \;=\; \mathrm{cap}^*(B,A). \tag{2.4}$$

As in the reversible case, the capacity is a monotone function in each of its coordinates:

Lemma 2.2. *Fix two disjoint subsets A, B of E. Consider two sets A', B' such that $A\subset A'\subset B^c$ and $B\subset B'\subset A^c$. Then,*

$$\mathrm{cap}(A,B) \;\le\; \mathrm{cap}(A,B')\;, \quad \mathrm{cap}(A,B) \;\le\; \mathrm{cap}(A',B)\;.$$

Proof. The first claim follows from the original definition and the second one from equation (2.4). □

For two disjoint subsets A, B of E, let $V_{A,B}$, $V^*_{A,B} : E \to [0,1]$ be the equilibrium potentials defined by

$$V_{A,B}(x) \;=\; \mathbb{P}_x[T_A < T_B]\;, \quad V^*_{A,B}(x) \;=\; \mathbb{P}^*_x[T_A < T_B]\;. \tag{2.5}$$

When the set B^c is finite, the equilibrium potential $V_{A,B}$ has a finite support and belongs therefore to the domain of the generator. Moreover, in this case, $V_{A,B}$ is the unique solution of the elliptic equation

$$\begin{cases} (LV)(z) \;=\; 0\;, & z\in E\setminus(A\cup B)\;,\\ V(x) = 1\;, & x\in A\;,\\ V(y) = 0\;, & y\in B\;. \end{cases}$$

Furthermore, since by the Markov property, $-(LV_{A,B})(x) = \lambda(x)\mathbb{P}_x[T_B < T_A^+]$, $x\in A$,

$$\mathrm{cap}(A,B) \;=\; \langle V_{A,B}\,,\, (-L)\, V_{A,B}\rangle_\mu \;=\; D(V_{A,B})\;. \tag{2.6}$$

This identity does not hold in general, since the scalar product is not well defined if the set B^c is not finite. However, following [2], if the process X_t is positive recurrent and the measure $M(x) = \mu(x)\lambda(x)$ is finite, one can show that this formula for the capacity holds.

Lemma 2.3. *For any disjoints subsets A and B of E,*

$$\mathrm{cap}(A,B) \;=\; \mathrm{cap}(B,A)\;.$$

Moreover, if $\{K_n \,|\, n \ge 1\}$ *is an increasing sequence of finite sets such that* $E = \cup_{n\ge 1} K_n$*, then*

$$\mathrm{cap}(A,B) \;=\; \lim_{m\to+\infty} \lim_{n\to+\infty} \mathrm{cap}(A_m,B_n)\;,$$

where $A_m = A\cap K_m$*,* $B_n = B\cup K_n^c$*.*

Proof. Assume first that B^c is finite. In this case by (2.6),

$$\mathrm{cap}(A,B) \;=\; D(V_{A,B}) \;=\; D(1-V_{B,A}) \;=\; D(V_{B,A})\;.$$

Since $\sum_{x\in B, y\in B^c}\mu(x) r(x,y)$ and $\sum_{y\in B, x\in B^c}\mu(x) r(x,y)$ are finite, and since $V_{B,A}$ is equal to 1 on B, we may write $D(V_{B,A})$ as

$$\begin{aligned}
&\frac12 \sum_{x,y} \mu(x) r(x,y) V_{B,A}(x) (V_{B,A}(x) - V_{B,A}(y))\\
&\quad +\; \frac12 \sum_{x,y} \mu(y) r(y,x) V_{B,A}(y) (V_{B,A}(y) - V_{B,A}(x))\\
&\quad +\; \sum_{x,y} c_a(x,y) V_{B,A}(y) (V_{B,A}(y) - V_{B,A}(x))\;,
\end{aligned} \tag{2.7}$$

where $c_a(x,y) = (1/2)[\mu(x) r(x,y) - \mu(y) r(y,x)]$ and all these sums are absolutely convergent since $V_{B,A}$ is bounded.

The first two lines of the previous sum are equal. Since $V_{B,A}$ is bounded, $(LV_{B,A})(x)$ is well defined by (2.1) for each x in E. Moreover, $(LV_{B,A})(x) = 0$ for $x\in (A\cup B)^c$ and $-(LV_{B,A})(x) = \lambda(x)\mathbb{P}_x[T_A < T_B^+]$, $x\in B$. Therefore, the sum of the first two lines is equal to

$$\sum_x \mu(x) V_{B,A}(x) (-LV_{B,A})(x) \;=\; \sum_{b\in B} M(b)\, \mathbb{P}_b [T^+_A < T^+_B] \;=\; \mathrm{cap}(B,A)\;. \tag{2.8}$$

On the other hand, since $c_a(x,y) = - c_a(y,x)$ and since the sum $\sum_{x,y} u_{x,y}$ may be written as $(1/2) \sum_{x,y} \{u_{x,y} + u_{y,x}\}$, the last line in (2.7) is equal to

$$\begin{aligned}
&\frac12 \sum_{x,y} c_a(x,y) (V^2_{B,A}(y) - V^2_{B,A}(x))\\
&\quad =\; \frac12 \sum_{x,y\not\in B} c_a(x,y) (V^2_{B,A}(y) - V^2_{B,A}(x)) + \sum_{x\not\in B}\sum_{y\in B} c_a(x,y) (1 - V^2_{B,A}(x))\\
&\quad =\; -\sum_{x\not\in B}\sum_{y\in E} c_a(x,y) V^2_{B,A}(x) + \sum_{x\not\in B}\sum_{y\in B} c_a(x,y)\;.
\end{aligned}$$

As $c_a(x,y) = -c_a(y,x)$, $\sum_{x,y\not\in B} c_a(x,y) = 0$. We may therefore replace the sum over B in the last term by a sum over E. Since μ is a stationary state, $\sum_{y\in E} c_a(x,y) = 0$ for all $x\in E$. This proves that the last line of the previous displayed formula vanishes. In conclusion, when B^c is finite,

$$\mathrm{cap}(A,B) \;=\; D(V_{B,A}) \;=\; \mathrm{cap}(B,A)\;.$$

It remains to remove the assumption that B^c is finite. Let $\{K_n \,|\, n\ge 1\}$ be an increasing sequence of finite sets such that $E = \cup_{n\ge1} K_n$. For each $m\le n$, let $A_m = A\cap K_m$, $B_n = B\cup K_n^c$ and note that B_n^c is finite for each $n\ge 1$. Since each set A_m is finite, by (2.3), by (2.4) and by Beppo Levi's theorem,

$$\begin{aligned}\lim_{m\to+\infty}\lim_{n\to+\infty}\mathrm{cap}(A_m,B_n) \;&=\; \lim_{m\to+\infty}\mathrm{cap}(A_m,B) \;=\; \lim_{m\to+\infty}\mathrm{cap}^*(B,A_m)\\ &=\; \lim_{m\to+\infty}\sum_{b\in B} M(b)\,\mathbb{P}^*_b[T^+_B > T^+_{A_m}] \;=\; \mathrm{cap}^*(B,A) \;=\; \mathrm{cap}(A,B)\;.\end{aligned}$$

Since B_n^c is finite, by (2.4) and by the first part of the proof, for any $m\le n$, $\mathrm{cap}(A_m,B_n)=\mathrm{cap}(B_n,A_m)=\mathrm{cap}^*(A_m,B_n)$. Repeating the same computations we obtain that

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\mathrm{cap}(A_m,B_n) \;=\; \lim_{m\to+\infty}\lim_{n\to+\infty}\mathrm{cap}^*(A_m,B_n) \;=\; \mathrm{cap}(B,A)\;.$$

This proves the lemma. □

Denote by S (resp. A) the symmetric (resp. anti-symmetric) part of the generator L in $L^2(\mu)$: $S=(1/2)\{L+L^*\}$, $A=(1/2)\{L-L^*\}$. The next result is proved in Section 3.

Theorem 2.4. *Fix two disjoint subsets A, B of E, with B^c finite. Then,*

$$\mathrm{cap}(A,B) \;=\; \inf_F\sup_H\Big\{2\langle L^*F\,,\,H\rangle_\mu \,-\, \langle H,(-S)H\rangle_\mu\Big\}\;, \tag{2.9}$$

where the supremum is carried over all functions $H:E\to\mathbb{R}$ which are constant at A and B, and where the infimum is carried over all functions F which are equal to 1 at A and 0 at B. Moreover, the function $F_{A,B}$ which solves the variational problem for the capacity is equal to $(1/2)\{V_{A,B}+V^*_{A,B}\}$, where $V_{A,B}$, $V^*_{A,B}$ are the harmonic functions defined in (2.5).

In the reversible case, the supremum over H in the statement of Theorem 2.4 is easily shown to be equal to $\langle(-L)F\,,\,F\rangle_\mu$ and we recover the well known variational formula for the capacity:

$$\mathrm{cap}(A,B) \;=\; \inf_F\langle(-L)F\,,\,F\rangle_\mu\;,$$

where the infimum is carried over all functions F which are equal to 1 at A and 0 at B.

When the set E is finite and the sets A and B are singletons, $A=\{a\}$, $B=\{b\}$, the supremum over H becomes a supremum over all functions $H:E\to\mathbb{R}$. In this case,

$$\sup_H\Big\{2\langle L^*F\,,\,H\rangle_\mu \,-\, \langle H,(-S)H\rangle_\mu\Big\} \;=\; \langle L^*F\,,\,(-S)^{-1}L^*F\rangle_\mu\;. \tag{2.10}$$

Therefore, when the set E is finite, since $L\,(-S)^{-1}L^*=\{[(-L)^{-1}]^s\}^{-1}$ [12, Section 2.5], the formula for the capacity between two singletons becomes

$$\mathrm{cap}(\{a\},\{b\}) \;=\; \inf_F\langle F\,,\,L(-S)^{-1}L^*F\rangle_\mu \;=\; \inf_F\langle F\,,\,\{[(-L)^{-1}]^s\}^{-1}F\rangle_\mu\;,$$

where the infimum is carried over all functions F which are equal to 1 at a and 0 at b, and where $[(-L)^{-1}]^s$ stands for the symmetric part of the operator $(-L)^{-1}$.

In Lemma 4.1 below we express the right hand side of (2.10) as an infimum over divergence free flows. We have therefore two alternative formulas for the scalar product $\langle L^*F\,,\,(-S)^{-1}L^*F\rangle_\mu$, one expressed as a supremum over functions, and another one written as an infimum over flows.

2.1. **Estimates on the capacity.** We compare in this subsection the capacity associated to the generator L with the symmetric capacities cap^s associated to the generators S. Let $\{X^s_t : t \ge 0\}$ be the Markov process on E with generator S. We shall refer to X^s_t as the symmetric or reversible version of the process X_t. Denote by $\mathbb{P}^s_x$, $x \in E$, the probability measure on the path space $D(\mathbb{R}_+, E)$ induced by the Markov process X^s_t starting from x.

For two disjoint subsets A, B of E, let $\mathrm{cap}^s(A,B)$ be the capacity between the sets A and B for the reversible process X^s_t:

$$\mathrm{cap}^s(A,B) \;=\; \sum_{x\in A} M(x)\, \mathbb{P}^s_x\big[T^+_A > T^+_B\big] \;.$$

In the case where the set B^c is finite,

$$\mathrm{cap}^s(A,B) \;=\; \langle V^s_{A,B} \,,\, (-S)\, V^s_{A,B}\rangle_\mu \;,$$

where $V^s_{A,B}$ is the equilibrium potential: $V^s_{A,B}(x) \;=\; \mathbb{P}^s_x[T_A < T_B]$. Moreover, since the generator S is symmetric in $L^2(\mu)$, it is well known that if B^c is finite,

$$\mathrm{cap}^s(A,B) \;=\; \inf_F \langle (-S)F \,,\, F\rangle_\mu \;=\; \inf_F \langle (-L)F \,,\, F\rangle_\mu \;, \tag{2.11}$$

where the infimum is carried over all functions F which are equal to 1 at A and 0 at B. Taking $H = -F$ in the variational formula (2.9) we obtain that

$$\mathrm{cap}(A,B) \;\ge\; \inf_F \langle (-L)F \,,\, F\rangle_\mu \;.$$

The next result follows from the previous observation, (2.11) and Lemma 2.3.

Lemma 2.5. *For two disjoint subsets A, B of E,*

$$\mathrm{cap}^s(A,B) \;\le\; \mathrm{cap}(A,B) \;.$$

Recall that a generator L satisfies a sector condition with constant C_0 if for every f, g in the domain of the generator,

$$\langle Lf, g\rangle^2_\mu \;\le\; C_0\, \langle (-L)f, f\rangle_\mu\, \langle (-L)g, g\rangle_\mu \;.$$

Next result, whose proof is presented at the end of Section 3, shows that if the generator L satisfies a sector condition, we may estimate the capacity between two sets by the capacity associated to the symmetric part of the generator.

Lemma 2.6. *Suppose that the generator L satisfies a sector condition with constant C_0. Then, for every pair of disjoint subsets A, B of E,*

$$\mathrm{cap}(A,B) \;\le\; C_0\, \mathrm{cap}^s(A,B) \;.$$

2.2. **Flows in finite state spaces.** We have seen that one can reduce capacity computations to the case when B^c is finite. By identifying B with a single point (this will be rigorously done in the next section) we can then restrict ourselves to the case of a finite space E. We then assume in this subsection that the E is *finite*. In this case the stationary measure μ is unique up to multiplicative constants. Define the (generally asymmetric) conductances

$$c(x,y) \;=\; \mu(x)\, r(x,y) \;, \quad c^*(x,y) \;=\; \mu(x)\, r^*(x,y) \;, \quad x \neq y \in E \;.$$

Note that $c(x,y) = c^*(y,x)$. Let $c_s(x,y)$, $c_a(x,y)$, be the symmetric and the asymmetric parts of the conductances:

$$c_s(x,y) \;=\; (1/2)\{c(x,y) + c^*(x,y)\} \;, \quad c_a(x,y) \;=\; (1/2)\{c(x,y) - c^*(x,y)\} \;,$$

for $x \neq y \in E$. Clearly, $c_s(x,y) = c_s(y,x)$ and $c_a(x,y) = -c_a(y,x)$. The symmetric conductances $c_s(x,y)$ are the conductances of the reversible Markov process associated to the generator S.

Denote by $\mathcal{E}$ the set of oriented edges or arcs of E: $\mathcal{E} = \{(x,y) \in E \times E : c_s(x,y) > 0\}$. For an oriented edge $e = (x,y) \in \mathcal{E}$, let $e^- = x$ be the tail of the arc e and let $e^+ = y$ be its head. We call *flow* any anti-symmetric function $\varphi : \mathcal{E} \to \mathbb{R}$. Denote by $\mathcal{F}$ the set of flows endowed with the scalar product

$$\langle \varphi, \psi \rangle \;=\; \frac{1}{2} \sum_{(x,y)\in\mathcal{E}} \frac{1}{c_s(x,y)}\, \varphi(x,y)\, \psi(x,y)\;, \tag{2.12}$$

and let $\| \cdot \|$ be the norm associated to this scalar product.

Denote by $(\mathrm{div}\ \varphi)(x)$, $x \in E$, the *divergence* of the flow φ at x:

$$(\mathrm{div}\ \varphi)(x) \;=\; \sum_{y:(x,y)\in\mathcal{E}} \varphi(x,y)\;, \quad x \in E\;.$$

A flow φ whose divergence vanishes at all sites, $(\mathrm{div}\varphi)(x) = 0$ for all $x \in E$, is called a *divergence free* flow. An important example of such a divergence free flow is c_a.

For a function $f : E \to \mathbb{R}$, let $\Psi_f(x,y) = c_s(x,y)[f(x) - f(y)]$ be the gradient flow associated to f. Clearly, Ψ_f belongs to $\mathcal{F}$ and

$$\|\Psi_f\|^2 \;=\; \langle \Psi_f, \Psi_f \rangle \;=\; \langle (-L)f\,,\, f\rangle_\mu\;. \tag{2.13}$$

Let $\mathcal{G} = \{\Psi_f \,|\, f : E \to \mathbb{R}\} \subset \mathcal{F}$. We refer to $\mathcal{G}$ as the set of *gradient flows*.

A finite sequence $\gamma = (x_0, \dots, x_n = x_0)$ of sites in E which starts and ends at the same site is called a *cycle* if (x_i, x_{i+1}) is an arc for each $0 \le i \le n-1$ and if $x_i \neq x_j$ for $0 \le i < j < n$. An arc e is said to belong to a cycle $\gamma = (x_0, \dots, x_n = x_0)$ if $e = (x_i, x_{i+1})$ for some $0 \le i < n$. We associate to a cycle $\gamma = (x_0, \dots, x_n = x_0)$ the flow $\chi_\gamma : \mathcal{E} \to \mathbb{R}$ defined by

$$\chi_\gamma \;=\; \sum_{i=0}^{n-1} \{\delta_{(x_i,x_{i+1})} - \delta_{(x_{i+1},x_i)}\}\;. \tag{2.14}$$

Denote by $\mathcal{C}$ the subspace of $\mathcal{F}$ spanned by flows associated to cycles.

A flow $\varphi \in \mathcal{C}$ associated to a cycle has no divergence:

$$(\mathrm{div}\ \varphi)(x) \;=\; 0 \quad x \in E\;.$$

Also, φ is a gradient flow if and only if φ is orthogonal to all cycle flows. In other words, we have

$$\mathcal{F} = \mathcal{G} \oplus \mathcal{C}\;, \quad \mathcal{G} \perp \mathcal{C}\;. \tag{2.15}$$

In addition, a flow φ that is orthogonal to all gradient flows satisfies, $(\mathrm{div}\varphi)(x_0) = 0$ for all x_0 in E, because $(\mathrm{div}\varphi)(x_0) = \langle \Psi_f, \varphi\rangle$ for the function f defined by $f(x) = \delta_{x_0,x}$. This proves that $\mathcal{C}$ is the set of all divergence free flows.

Inspired by the computation of the current through an arc (x,y), presented in (4.5) below, for a function $f : E \to \mathbb{R}$, denote by $\Phi_f : \mathcal{E} \to \mathbb{R}$ the flow defined by

$$\Phi_f(x,y) \;=\; f(x)\, c(x,y) \;-\; f(y)\, c(y,x)\;. \tag{2.16}$$

In Section 4 we prove the following result.

Theorem 2.7. *For any disjoint and non-empty sets A, $B \subset E$,*

$$\mathrm{cap}(A,B) \;=\; \inf_f \, \inf_\varphi \, \|\Phi_f - \varphi\|^2 \;,$$

where the first infimum is carried over all functions $f : E \to \mathbb{R}$ which are equal to 1 on the set A and 0 on the set B, and the second infimum is carried over all flows $\varphi \in \mathcal{F}$ such that

$$(\mathrm{div}\ \varphi)(x) = 0 \;, \quad x \in (A\cup B)^c \;, \quad \sum_{a\in A} (\mathrm{div}\ \varphi)(a) = 0 \;, \quad \sum_{b\in B} (\mathrm{div}\ \varphi)(b) = 0 \;.$$

In the case where A and B are singletons, the second infimum is carried over all divergence free flows. Hence, in the case of singletons, the infimum corresponds to a projection over the space of gradient flows.

In the last section of this article, when we shall estimate some capacities among singletons, the divergence free flow $\varphi(x,y) = c_a(x,y)$, $(x,y) \in \mathcal{E}$, will be used repeatedly to obtain upper bounds.

2.3. **Transient Markov processes.** Assume in this subsection that the irreducible Markov process $\{X_t \,|\, t\ge 0\}$ is transient, and denote by $G(x,y)$ its Green function:

$$G(x,y) \;=\; \mathbb{E}_x\Big[\int_0^\infty \mathbf{1}\{X_t = y\}\, dt \Big] \;.$$

Define the capacity of a state $x\in E$, denoted by $\mathrm{cap}(x)$, as

$$\mathrm{cap}(x) \;=\; M(x)\, \mathbb{P}_x\big[T_x^+ = \infty\big] \;.$$

Since $G(x,x)^{-1} = \lambda(x)\mathbb{P}_x[T^+_x = \infty]$, we have that

$$\mathrm{cap}(x) \;=\; \mu(x)\, \frac{1}{G(x,x)} \;\cdot \tag{2.17}$$

Fix a finitely supported function $f : E\to\mathbb{R}$ such that $f(x)\not = 0$, and let $F(y) = f(y)/f(x)$ so that $F(x)=1$. By Definition 2.1, if $\{A_n \,|\, n\ge 1\}$ is a sequence of increasing, finite sets such that $E = \cup_{n\ge 1} A_n$,

$$\mathrm{cap}(x) \;=\; \lim_{n\to\infty} \mathrm{cap}(x, A_n^c) \;.$$

Since A_n is finite and since F is finitely supported, with $F(x)=1$, by Theorem 2.4,

$$\mathrm{cap}(x) \;\le\; \lim_{n\to\infty} \sup_{H\in \mathfrak{B}_n} \Big\{ 2 \langle L^* F \,,\, H\rangle_\mu \,-\, \langle H, (-S) H\rangle_\mu \Big\} \;,$$

where $\mathfrak{B}_n$ is the set of functions $H : E\to \mathbb{R}$ which vanish at A_n^c. As $F(\,\cdot\,) = f(\,\cdot\,)/f(x)$, replacing H by $H'(\,\cdot\,) = H(\,\cdot\,)/f(x)$, we obtain that

$$\mathrm{cap}(x) \;\le\; \frac{1}{f(x)^2} \lim_{n\to\infty} \sup_{H\in \mathfrak{B}_n} \Big\{ 2 \langle L^* f \,,\, H\rangle_\mu \,-\, \langle H, (-S) H\rangle_\mu \Big\} \;.$$

In view of (2.17), we have proved the following result, which generalizes a well known estimate in the context of reversible Markov processes [12, Proposition 5.23], [1, Lemma 2.1].

Lemma 2.8. *Let $f : E\to\mathbb{R}$ be a finitely supported function and let $\{A_n \,|\, n\ge 1\}$ be a sequence of increasing, finite sets such that $E = \cup_{n\ge 1} A_n$. Then, for every $x\in E$,*

$$\mu(x)\, f(x)^2 \;\le\; G(x,x) \, \lim_{n\to\infty} \sup_{H\in \mathfrak{B}_n} \Big\{ 2 \langle L^* f \,,\, H\rangle_\mu \,-\, \langle H, (-S) H\rangle_\mu \Big\} \;,$$

where $\mathfrak{B}_n$ is the set of functions $H : E \to \mathbb{R}$ which vanish at A_n^c.

3. Collapsed Chains and Proof of Theorem 2.4

We start this section by assuming that E is *finite* and that μ is the unique stationary probability measure. In the case where the sets A and B are singletons, the proof of Theorem 2.4 takes the following form.

Lemma 3.1. *Fix a pair of points $a \neq b$ in a finite set E. Then,*

$$\operatorname{cap}(\{a\},\{b\}) \;=\; \inf_f \langle f\,,\, L\,(-S)^{-1}L^* f\rangle_\mu\;, \tag{3.1}$$

where the infimum is carried over all function $f : E \to \mathbb{R}$ such that $f(a) = 1$, $f(b) = 0$. Moreover, the function $f_{a,b}$ which solves the variational problem (3.1) is unique and equal to $(1/2)\{V_{a,b} + V^*_{a,b}\}$, where $V_{a,b}$, $V^*_{a,b}$ are the harmonic functions defined in (2.5).

Proof. The operator $L(-S)^{-1}L^*$ restricted to the space of mean zero functions is symmetric, strictly positive and bounded because the state space is finite. There exists, in particular, a unique function $f_{a,b}$ which solves the variational problem (3.1). Moreover, as $(-S)^{-1}$ is also strictly positive on the the space of mean zero functions, there exists a strictly positive constant C_0 such that

$$\langle f_{a,b}\,,\, L\,(-S)^{-1}L^* f_{a,b}\rangle_\mu \;\ge\; C_0 \langle L^* f_{a,b}\,,\, L^* f_{a,b}\rangle_\mu \;>\; 0\;. \tag{3.2}$$

The previous expression can not vanish due to the boundary conditions of $f_{a,b}$.

Since $f_{a,b}$ solves the variational problem (3.1),

$$(L\,(-S)^{-1}L^* f_{a,b})(x) \;=\; 0\;, \quad x \neq a, b\;.$$

Let $W_{a,b} = S^{-1}L^* f_{a,b} + c_0$, where c_0 is a constant chosen for $W_{a,b}$ to vanish at b: $W_{a,b}(b) = 0$. Since $LW_{a,b} = 0$ on $E \setminus \{a, b\}$, $W_{a,b}$ is a multiple of the harmonic function $V_{a,b}$ introduced in (2.5): $W_{a,b} = \lambda V_{a,b}$, where $\lambda = W_{a,b}(a)$.

We claim that $\lambda = 1$ so that $W_{a,b} = V_{a,b}$. Indeed, since $W_{a,b}$ is harmonic on $E \setminus \{a, b\}$ and $f_{a,b}(a) = 1$, $f_{a,b}(b) = 0$,

$$\langle f_{a,b}\,,\, L\,(-S)^{-1}L^* f_{a,b}\rangle_\mu \;=\; \langle f_{a,b}\,,\, (-L)\, W_{a,b}\rangle_\mu \;=\; -\mu(a)\,(L\, W_{a,b})(a)\;.$$

On the other hand, since $W_{a,b} - c_0 = S^{-1}L^* f_{a,b}$ and SS^{-1} is the identity,

$$\begin{aligned}\langle f_{a,b}\,,\, L\,(-S)^{-1}L^* f_{a,b}\rangle_\mu \;&=\; \langle L^* f_{a,b}\,,\, (-S)^{-1}L^* f_{a,b}\rangle_\mu \;=\; \langle W_{a,b}\,,\, (-S)W_{a,b}\rangle_\mu \\ &=\; \langle W_{a,b}\,,\, (-L)W_{a,b}\rangle_\mu \;=\; -\mu(a)\, W_{a,b}(a)\,(L\, W_{a,b})(a)\;,\end{aligned}$$

where the last identity follows from the fact that $W_{a,b}$ is harmonic on $E \setminus \{a, b\}$ and that $W_{a,b}(b) = 0$. By (3.2) and the two previous displayed formulas, $W_{a,b}(a) = 1$ so that $W_{a,b} = V_{a,b}$. Hence, by the last displayed formula and (2.3),

$$\langle f_{a,b}\,,\, L\,(-S)^{-1}L^* f_{a,b}\rangle_\mu \;=\; \langle V_{a,b}\,,\, (-L)V_{a,b}\rangle_\mu \;=\; \operatorname{cap}(\{a\},\{b\})\;,$$

which concludes the proof of the first assertion of the lemma.

Denote by $f_{a,b}$ a function which solves the variational problem (3.1). We claim that $f_{a,b} = (1/2)\{V_{a,b} + V^*_{a,b}\}$. Indeed, since $W_{a,b} = V_{a,b}$ and since $V_{a,b}$ is L-harmonic on $E \setminus \{a, b\}$, on this set $(1/2)L^*V_{a,b} = SV_{a,b} = L^* f_{a,b}$. Furthermore, as $V^*_{a,b}$ is L^*-harmonic on $E \setminus \{a, b\}$, we have in fact that $(1/2)L^*\{V_{a,b} + V^*_{a,b}\} = L^* f_{a,b}$. Hence,

$$\begin{gathered} L^*(1/2)\{V_{a,b} + V^*_{a,b}\} \;=\; L^* f_{a,b} \text{ on } E \setminus \{a, b\} \\ \text{and } (1/2)\{V_{a,b} + V^*_{a,b}\} \;=\; f_{a,b} \text{ on } \{a, b\}\;. \end{gathered}$$

It follows from these two identities that $f_{a,b} = (1/2)\{V_{a,b} + V^*_{a,b}\}$. □

To extend the previous result to the case where the sets A and B are not singletons, we define a Markov chain in which a set is collapsed to a single state. Fix a subset A of E, and let $\overline{E}_A = [E \setminus A] \cup \{\mathfrak{d}\}$, where $\mathfrak{d}$ is an extra site added to E to represent the collapsed set A. Denote by $\{\overline{X}^A_t : t \ge 0\}$ the chain obtained from X_t by collapsing the set A to a singleton. This is the Markov process on $\overline{E}_A$ with jump rates $\overline{r}_A(x,y)$, x, $y \in \overline{E}_A$, given by

$$\begin{aligned} \overline{r}_A(x,y) \;&=\; r(x,y)\;, \quad \overline{r}_A(x,\mathfrak{d}) \;=\; \sum_{z\in A} r(x,z)\;, \quad x,y\in E\setminus A\;, \\ \overline{r}_A(\mathfrak{d},x) \;&=\; \frac{1}{\mu(A)}\sum_{y\in A}\mu(y)\, r(y,x)\;, \quad x\in E\setminus A\;. \end{aligned} \tag{3.3}$$

The collapsed chain $\{\overline{X}^A_t : t \ge 0\}$ inherits the irreducibility from the original chain.

Denote by $\overline{\mu}_A$ the probability measure on $\overline{E}_A$ given by

$$\overline{\mu}_A(\mathfrak{d}) \;=\; \mu(A)\;, \quad \overline{\mu}_A(x) \;=\; \mu(x)\;, \quad x\in E\setminus A\;. \tag{3.4}$$

Since

$$\sum_{y\notin A, z\in A} c(y,z) \;=\; \sum_{y\notin A, z\in A} c(z,y)\;,$$

one checks that $\overline{\mu}_A$ is a stationary state, and therefore the unique invariant probability measure, for the collapsed chain $\overline{X}^A_t$.

We may extend the concept of collapsed chain to the case in which more than one set is collapsed to a singleton. One can proceed recursively, collapsing first a set A to a point $a \notin E$, obtaining a Markov chain in $(E \setminus A) \cup \{a\}$, and then collapsing a set $B \subset E$, $B \cap A = \varnothing$, to a point $b \notin E \cup \{a\}$, obtaining a new Markov chain in $[E \setminus (A \cup B)] \cup \{a, b\}$. One checks that the final process is the same if we first collapse B and then A. The rate $\overline{r}_{A,B}(a, b)$ at which the collapsed chain jumps from a to b is given by

$$\overline{r}_{A,B}(a,b) \;=\; \frac{1}{\mu(A)}\sum_{z\in A}\mu(z)\sum_{x\in B} r(z,x)\;. \tag{3.5}$$

Denote by $\overline{L}_A$ the generator of the chain $\{\overline{X}^A_t : t \ge 0\}$ and by $\overline{L}^*_A$ the adjoint of $\overline{L}_A$ in $L^2(\overline{\mu}_A)$. Recall that we represent by $\{X^*_t : t \ge 0\}$ the adjoint of the chain X_t and by L^* its generator. Let $\{\overline{X^*}^A_t : t \ge 0\}$ be the chain obtained from X^*_t by collapsing the set A to a singleton and by $\overline{L^*}_A$ the generator of this process. We claim that

$$\overline{L}^*_A \;=\; \overline{L^*}_A\;. \tag{3.6}$$

To prove this claim, denote by $\overline{r}^*_A(x,y)$ the rates of the adjoint of $\{\overline{X}^A_t : t \ge 0\}$:

$$\overline{r}^*_A(x,y) \;=\; \frac{\overline{\mu}_A(y)\,\overline{r}_A(y,x)}{\overline{\mu}_A(x)}\;, \quad x\,,\, y\in \overline{E}_A\;.$$

Let $r^*(x,y)$, x, $y \in E$, be the jump rates of the adjoint process and let $\overline{r^*}_A(x,y)$, x, $y \in \overline{E}_A$, be the jump rates of its collapsed version.

In view of the previous displayed formula and by (3.3), (3.4), for x, $y \in E \setminus A$,

$$\overline{r}^*_A(x,y) \;=\; \frac{\mu(y)\, r(y,x)}{\mu(x)} \;=\; r^*(y,x) \;=\; \overline{r^*}_A(x,y)\;.$$

Furthermore, for $y \in E \setminus A$, since $\overline{\mu}_A(\mathfrak{d}) = \mu(A)$, by (3.3),

$$\overline{r}^*_A(\mathfrak{d}, y) \;=\; \frac{\mu(y)\, \overline{r}_A(y,\mathfrak{d})}{\mu(A)} \;=\; \frac{\mu(y) \sum_{z\in A} r(y,z)}{\mu(A)} \;=\; \frac{\sum_{z\in A} \mu(z)\, r^*(z,y)}{\mu(A)} \;=\; \overline{r^*}_A(\mathfrak{d}, y)\;.$$

Finally, for $x \in E \setminus A$, by analogous reasons,

$$\overline{r}^*_A(x, \mathfrak{d}) \;=\; \frac{\mu(A)\, \overline{r}_A(\mathfrak{d},x)}{\mu(x)} \;=\; \frac{\sum_{z\in A} \mu(z) r(z,x)}{\mu(x)} \;=\; \sum_{z\in A} r^*(x,z) \;=\; \overline{r^*}_A(x, \mathfrak{d})\;,$$

which proves (3.6). It follows from this result that

$$\overline{S}_A \;=\; (1/2)\big\{\overline{L}_A + \overline{L}_A^*\big\}\;, \tag{3.7}$$

if $\overline{S}_A$ stands for the generator $S = (1/2)(L + L^*)$ collapsed on the set A.

Fix two functions f, $g : \overline{E}_A \to \mathbb{R}$. Let F, $G : E \to \mathbb{R}$ be defined by $F(x) = f(x)$, $x \in E \setminus A$, $F(z) = f(\mathfrak{d})$, $z \in A$, with a similar definition for G. We claim that

$$\langle \overline{L}_A f\,,\, g\rangle_{\overline{\mu}_A} \;=\; \langle LF\,,\, G\rangle_\mu\;. \tag{3.8}$$

Conversely, if F, $G : E \to \mathbb{R}$ are two functions constant over A, (3.8) holds if we define f, $g : \overline{E}_A \to \mathbb{R}$ by $f(x) = F(x)$, $x \in E \setminus A$, $f(\mathfrak{d}) = F(z)$ for some $z \in A$.

Fix two functions f, $g : \overline{E}_A \to \mathbb{R}$. By definition of $\overline{L}_A$,

$$\langle \overline{L}_A f\,,\, g\rangle_{\overline{\mu}_A} \;=\; \sum_{x,y\in \overline{E}_A} \overline{\mu}_A(x)\, \overline{r}_A(x,y)\, [f(y) - f(x)]\, g(x)\;.$$

In view of (3.3), (3.4), this expression is equal to

$$\sum_{x\in E\setminus A} \mu(x) \Big\{ \sum_{y\in E\setminus A} r(x,y)\,[f(y)-f(x)] + \sum_{z\in A} r(x,z)\,[f(\mathfrak{d})-f(x)] \Big\}\, g(x) + \sum_{y\in E\setminus A} \sum_{z\in A} \mu(z)\, r(z,y)\, [f(y) - f(\mathfrak{d})]\, g(\mathfrak{d})\;.$$

Since $F(x) = f(x)$ for $x \in E \setminus A$, and $F(y) = f(\mathfrak{d})$ for $y \in A$, with similar identities with G, g replacing F, f, the last sum is equal to

$$\sum_{x\in E\setminus A} \mu(x) \Big\{ \sum_{y\in E\setminus A} r(x,y)\,[F(y)-F(x)] + \sum_{z\in A} r(x,z)\,[F(z)-F(x)] \Big\}\, G(x) + \sum_{z\in A} \sum_{y\in E\setminus A} \mu(z)\, r(z,y)\, [F(y) - F(z)]\, G(z)\;.$$

Since F is constant on A, we may add to this expression

$$\sum_{x\in A} \sum_{y\in A} \mu(x)\, r(x,y)\, [F(y) - F(x)]\, G(x)$$

to obtain that the last displayed expression is equal to $\langle LF, G\rangle_\mu$, which concludes the proof of the first assertion of (3.8). The second statement is obtained following the computation in reverse order.

It follows from (3.6) and (3.8) that

$$\langle \overline{L^*}_A f\,,\, g\rangle_{\overline{\mu}_A} \;=\; \langle L^* F\,,\, G\rangle_\mu\;, \quad \langle \overline{S}_A f\,,\, g\rangle_{\overline{\mu}_A} \;=\; \langle SF\,,\, G\rangle_\mu\;. \tag{3.9}$$

The next assertion establishes the relation between collapsed chains and capacities. Fix two disjoint subsets A and B of E. Let $\overline{E}_{A,B} = [E \setminus (A \cup B)] \cup \{a, b\}$, where $a \neq b$ are states which do not belong to E. Denote by $\{\overline{X}_t^{A,B} : t \geq 0\}$ the chain in which the sets A, B have been collapsed to the states a, b. Let $\overline{r}_{A,B}(x,y)$, $\overline{p}_{A,B}(x,y)$, and $\overline{\lambda}_{A,B}(x)$, x, $y \in \overline{E}_{A,B}$, be the jump rates, the jump probabilities, and the holding rates, respectively, of the chain $\overline{X}_t^{A,B}$, and denote by $\overline{\mu}_{A,B}$ its unique invariant probability measure.

Denote by $\overline{\text{cap}}_{A,B}$ the capacity associated to the collapsed chain. We claim that

$$\overline{\text{cap}}_{A,B}(\{a\},\{b\}) \;=\; \text{cap}(A,B)\;. \tag{3.10}$$

Denote by $\overline{\mathbb{P}}_x^{A,B}$, $x \in \overline{E}_{A,B}$, the probability measure on $D(\mathbb{R}_+, \overline{E}_{A,B})$ induced by the collapsed chain $\overline{X}_t^{A,B}$ starting from x. By Definition 2.1,

$$\begin{aligned}\overline{\text{cap}}_{A,B}(\{a\},\{b\}) \;&=\; \overline{M}_{A,B}(a)\,\overline{\mathbb{P}}_a^{A,B}\big[\,T_a^+ > T_b^+\,\big] \\ &=\; \overline{M}_{A,B}(a) \sum_{x\in\overline{E}_{A,B}} \overline{p}_{A,B}(a,x)\,\overline{\mathbb{P}}_x^{A,B}\big[\,T_a > T_b\,\big]\;,\end{aligned}$$

where $\overline{M}_{A,B}(x) = \overline{\mu}_{A,B}(x)\overline{\lambda}_{A,B}(x)$. Since $\overline{M}_{A,B}(a)\,\overline{p}_{A,B}(a,x) = \overline{\mu}_{A,B}(a)\,\overline{r}_{A,B}(a,x)$ and since $\overline{p}_{A,B}(a,a) = 0$, by the explicit expression (3.3) for the rates of the collapsed chain, the previous expression is equal to

$$\begin{aligned}&\overline{\mu}_{A,B}(a) \sum_{x\in E\setminus[A\cup B]} \frac{1}{\mu(A)} \sum_{z\in A} \mu(z)\,r(z,x)\,\overline{\mathbb{P}}_x^{A,B}\big[\,T_a > T_b\,\big] \\ &\quad+\; \overline{\mu}_{A,B}(a)\,\overline{r}_{A,B}(a,b)\,\overline{\mathbb{P}}_b^{A,B}\big[\,T_a > T_b\,\big]\;.\end{aligned}$$

By construction, $\overline{\mathbb{P}}_x^{A,B}[\,T_a > T_b\,] = \mathbb{P}_x[\,T_A > T_B\,]$ for $x \in E\setminus[A\cup B]$, and $\overline{\mathbb{P}}_b^{A,B}[\,T_a > T_b\,] = 1 = \mathbb{P}_x[\,T_A > T_B\,]$, $x \in B$. Hence, as $\overline{\mu}_{A,B}(a) = \mu(A)$, by (3.5) the last sum is equal to

$$\sum_{x\in E\setminus A} \sum_{z\in A} \mu(z)\,r(z,x)\,\mathbb{P}_x\big[\,T_A > T_B\,\big]\;.$$

Since $\mathbb{P}_x[\,T_A > T_B\,] = 0$, $x \in A$, and since $\mu(z)\,r(z,x) = M(z)p(z,x)$, this expression is equal to

$$=\; \sum_{z\in A} M(z)\,\mathbb{P}_z\big[\,T_A^+ > T_B^+\,\big] \;=\; \text{cap}(A,B)\;,$$

which concludes the proof of claim (3.10).

Lemma 3.2. *Fix two disjoint subsets A, B of a finite set E. Then,*

$$\text{cap}(A,B) \;=\; \inf_F \sup_H \Big\{ 2\langle L^*F\,,\,H\rangle_\mu \;-\; \langle H, (-S)H\rangle_\mu \Big\}\;.$$

where the supremum is carried over all functions $H : E \to \mathbb{R}$ which are constant at A and B, and where the infimum is carried over all functions F which are equal to 1 at A and 0 at B. Moreover, the function $F_{A,B}$ which solves the variational problem for the capacity is equal to $(1/2)\{V_{A,B} + V_{A,B}^*\}$, where $V_{A,B}$, $V_{A,B}^*$ are the harmonic functions defined in (2.5).

Proof. Fix two disjoint subsets A, B of E. By Lemma 3.1 and identity (3.10),

$$\mathrm{cap}(A,B) \;=\; \inf_f \langle \overline{L}^*_{A,B} f\,,\; (-\mathcal{S})^{-1} \overline{L}^*_{A,B} f \rangle_{\overline{\mu}_{A,B}}\;, \tag{3.11}$$

where $\overline{L}_{A,B}$ is the generator of the chain $\{\overline{X}^{A,B}_t : t \ge 0\}$ introduced right after (3.9), $\mathcal{S}$ is the symmetric part of $\overline{L}_{A,B}$, $\mathcal{S} = (1/2)(\overline{L}_{A,B} + \overline{L}^*_{A,B})$, and where the infimum is carried over all function $f : \overline{E}_{A,B} \to \mathbb{R}$ such that $f(a) = 1$, $f(b) = 0$.

By the variational formula for the norm induced by the operator $(-\mathcal{S})^{-1}$, the previous expression is equal to

$$\inf_f \sup_h \Big\{ 2 \langle \overline{L}^*_{A,B} f\,,\, h\rangle_{\overline{\mu}_{A,B}} \;-\; \langle h, (-\mathcal{S}) h\rangle_{\overline{\mu}_{A,B}} \Big\}\;,$$

where the supremum is carried over all functions $h : \overline{E}_{A,B} \to \mathbb{R}$. By (3.6), $\overline{L}^*_{A,B} = \overline{L^*}_{A,B}$, and by (3.7), $\mathcal{S} = \overline{S}_{A,B}$, where $\overline{S}_{A,B}$ is the generator S collapsed at A and B. Hence, the previous displayed equation is equal to

$$\inf_f \sup_h \Big\{ 2 \langle \overline{L^*}_{A,B} f\,,\, h\rangle_{\overline{\mu}_{A,B}} \;-\; \langle h, (-\overline{S}_{A,B}) h\rangle_{\overline{\mu}_{A,B}} \Big\}\;.$$

Let F, $H : E \to \mathbb{R}$ be defined by $F(x) = f(x)$, $x \in E \setminus (A \cup B)$, $F(z) = f(a)$, $z \in A$, $F(y) = f(b)$, $y \in B$, with a similar definition for H. By (3.8), (3.9), the last variational problem can be rewritten as

$$\inf_F \sup_H \Big\{ 2 \langle L^* F\,,\, H\rangle_{\mu} \;-\; \langle H, (-S) H\rangle_{\mu} \Big\}\;. \tag{3.12}$$

where the supremum is carried over all functions $H : E \to \mathbb{R}$ which are constant at A and B, and where the infimum is carried over all functions F which are equal to 1 at A and 0 at B. This proves the first assertion of the lemma.

To prove the second assertion of the lemma, recall from Lemma 3.1 that

$$\mathrm{cap}(A,B) \;=\; \langle \overline{L}^*_{A,B} f_{a,b}\,,\; (-\mathcal{S})^{-1} \overline{L}^*_{A,B} f_{a,b} \rangle_{\overline{\mu}_{A,B}}\;,$$

where $f_{a,b} = (1/2)\{\overline{V}_{a,b} + \overline{V}^*_{a,b}\}$ and $\overline{V}_{a,b}$, $\overline{V}^*_{a,b}$ are the harmonic functions for the collapsed process. By the first part of the proof, the right hand side is equal to

$$\sup_H \Big\{ 2 \langle L^* F_{A,B}\,,\, H\rangle_{\mu} \;-\; \langle H, (-S) H\rangle_{\mu} \Big\}\;,$$

where the supremum is carried over all functions $H : E \to \mathbb{R}$ which are constant at A and B, and where $F_{A,B}(x) = f_{a,b}(x)$, $x \in E \setminus (A \cup B)$, $F_{A,B}(z) = 1$, $z \in A$, $F_{A,B}(y) = 0$, $y \in B$. As we have already seen, by construction of the collapsed process, for $x \in E \setminus (A \cup B)$,

$$\overline{V}_{a,b}(x) \;=\; \overline{\mathbb{P}}^{A,B}_x [\, T_a < T_b \,] \;=\; \mathbb{P}_x [\, T_A < T_B \,] \;=\; V_{A,B}(x)\;,$$

with a similar identity for $\overline{V}^*_{a,b}$. In conclusion,

$$\mathrm{cap}(A,B) \;=\; \sup_H \Big\{ 2 \langle L^* F_{A,B}\,,\, H\rangle_{\mu} \;-\; \langle H, (-S) H\rangle_{\mu} \Big\}\;,$$

where $F_{A,B} = (1/2)\{V_{A,B} + V^*_{A,B}\}$, concluding the proof of the lemma. □

Remark 3.3. *The expression inside braces in the displayed formula of Lemma 3.2 does not change if H is replaced by $H + c$, where c is a constant. We may therefore restrict the supremum to functions H which vanish at B.*

We finally turn to the case where E is denumerable. Fix two disjoint subsets A, B of E and suppose that $B^c \supset A$ is finite. Similarly to what we did earlier in this section, we define chain where the infinite set B is collapsed to a state.

Denote by $\overline{X}_t$ the Markov process on the finite set $B^c \cup \{\mathfrak{d}\}$, where $\mathfrak{d}$ is an extra site added to E to represent the collapsed, possibly infinite, set B, whose rates $\overline{r}(x,y)$, x, $y \in B^c \cup \{\mathfrak{d}\}$, are defined by

$$\begin{aligned} &\overline{r}(x,y) \;=\; r(x,y) \;, \quad \overline{r}(x,\mathfrak{d}) \;=\; \sum_{z\in B} r(x,z) \;, \quad x,y \in B^c \;, \\ &\overline{r}(\mathfrak{d},x) \;=\; \sum_{y\in B} \mu(y)\, r(y,x) \;, \quad x \in B^c \;. \end{aligned} \tag{3.13}$$

Note that $\overline{r}(\mathfrak{d},x)$ is finite because $\sum_{y\in E} \mu(y)\, r(y,x) = M(x) < \infty$, as μ is a stationary state, and that $\overline{r}(\mathfrak{d},x) > 0$ if there exists $z \in B$ such that $r(z,x) > 0$. In particular, the collapsed chain $\{\overline{X}_t : t \ge 0\}$ inherits the irreducibility from the original chain. Moreover, since $\sum_{x\in B^c} \sum_{y\in B} \mu(y) r(y,x) = \sum_{x\in B} \sum_{y\in B^c} \mu(y) r(y,x)$, $\overline{\mu}(x) = \mu(x)$, $x \in B^c$, $\overline{\mu}(\mathfrak{d}) = 1$ is a stationary measure.

Let $\overline{\mathbb{P}}_x$, $x \in B^c \cup \{\mathfrak{d}\}$, represent the probability measure on the path space $D(\mathbb{R}_+, B^c \cup \{\mathfrak{d}\})$ induced by the Markov process $\overline{X}_t$ starting from x. Clearly, for any $A \subset B^c$,

$$\mathbb{P}_y\big[T_B < T_A^+\big] \;=\; \overline{\mathbb{P}}_y\big[T_{\mathfrak{d}} < T_A^+\big] \;, \quad y \in B^c \;.$$

Therefore, by (2.3), for any $A \subset B^c$,

$$\mathrm{cap}(A,B) \;=\; \sum_{y\in A} M(y)\, \mathbb{P}_y\big[T_B < T_A^+\big] \;=\; \sum_{y\in A} \overline{M}(y)\, \overline{\mathbb{P}}_y\big[T_{\mathfrak{d}} < T_A^+\big] \;=\; \overline{\mathrm{cap}}(A,\mathfrak{d}) \;, \tag{3.14}$$

if $\overline{\mathrm{cap}}$ stands for the capacity of the collapsed chain.

Denote by $\overline{L}$ the generator of the collapsed chain. Fix a pair of functions f, $h : B^c \cup \{\mathfrak{d}\} \to \mathbb{R}$ such that $h(\mathfrak{d}) = 0$. Let F, $H : E \to \mathbb{R}$ be the functions defined by $F(x) = f(x)$, $x \in B^c$, $F(z) = f(\mathfrak{d})$, $z \in B$, with a similar definition for H. We claim that

$$\langle \overline{L} f, h\rangle_{\overline{\mu}} \;=\; \langle LF, H\rangle_{\mu} \;. \tag{3.15}$$

Conversely, if F, $H : E \to \mathbb{R}$ are constant on the set B and if H vanishes at B, (3.15) holds if $f, h : B^c \cup \{\mathfrak{d}\} \to \mathbb{R}$ are defined by $f(x) = F(x)$, $x \in B^c$, $f(\mathfrak{d}) = F(z)$, $z \in B$, with a similar definition for h.

To prove (3.15), fix a pair of functions f, $h : B^c \cup \{\mathfrak{d}\} \to \mathbb{R}$ with the above properties. By definition of the collapsed chain and since $h(\mathfrak{d}) = 0$, $\langle \overline{L} f, h\rangle_{\overline{\mu}}$ is equal to

$$\sum_{x,y\in B^c} \mu(x)\, r(x,y)\, h(x)\, [f(y) - f(x)] \;+\; \sum_{x\in B^c} \mu(x)\, \overline{r}(x,\mathfrak{d})\, h(x)\, [f(\mathfrak{d}) - f(x)] \;.$$

Since $\overline{r}(x,\mathfrak{d}) = \sum_{z\in B} r(x,z)$ and since F is constant over B, the second term is equal to

$$\sum_{x\in B^c} \sum_{y\in B} \mu(x)\, r(x,y)\, h(x)\, [f(\mathfrak{d}) - f(x)] \;=\; \sum_{x\in B^c} \sum_{y\in B} \mu(x)\, r(x,y)\, H(x)\, [F(y) - F(x)] \;.$$

Hence, adding the two terms,

$$\langle \overline{L} f, h\rangle_{\overline{\mu}} \;=\; \sum_{x\in B^c} \sum_{y\in E} \mu(x)\, r(x,y)\, H(x)\, [F(y) - F(x)] \;=\; \langle LF, H\rangle_{\mu}$$

because H vanishes on B. This proves the first assertion of claim. The converse one is proved by following the previous computation in the reverse order.

Proof of Theorem 2.4. Fix two disjoint subsets A, B of E and assume that B^c is finite. By (3.14), $\text{cap}(A,B) = \overline{\text{cap}}(A,\mathfrak{d})$. On the other hand, and by Lemma 3.2 and by Remark 3.3,

$$\overline{\text{cap}}(A,\mathfrak{d}) \;=\; \inf_f \sup_h \Big\{ 2\langle f\,,\, \overline{L}h\rangle_{\overline{\mu}} \,-\, \langle h, (-\overline{L})h\rangle_{\overline{\mu}} \Big\}\;,$$

where the supremum is carried over all functions $h : B^c\cup\{\mathfrak{d}\} \to \mathbb{R}$ which are constant at A and vanish at $\mathfrak{d}$, and where the infimum is carried over all functions f which are equal to 1 at A and 0 at $\mathfrak{d}$. Since f vanishes at $\mathfrak{d}$, by claim (3.15), the right hand side of the previous is equal to

$$\inf_F \sup_H \Big\{ 2\langle L^* F\,,\, H\rangle_{\mu} \,-\, \langle H, (-L)H\rangle_{\mu} \Big\}$$

where the supremum is carried over all functions $H : E \to \mathbb{R}$ which are constant at A and vanish at B, and where the infimum is carried over all functions F which are equal to 1 at A and 0 at B. The expression inside braces in the previous formula remains unchanged if we replace H by $H + c$, where c is a constant. We may therefore veil the assumption that H vanishes at B. This proves the first assertion of Theorem 2.4.

By (3.14) and by Lemma 3.2,

$$\text{cap}(A,B) \;=\; \overline{\text{cap}}(A,\mathfrak{d}) \;=\; \sup_h \Big\{ 2\langle f_{A,\mathfrak{d}}\,,\, \overline{L}h\rangle_{\overline{\mu}} \,-\, \langle h, (-\overline{L})h\rangle_{\overline{\mu}} \Big\}\;,$$

where $f_{A,\mathfrak{d}} = (1/2)\{V_{A,\mathfrak{d}} + V^*_{A,\mathfrak{d}}\}$, and $V_{A,\mathfrak{d}}$, $V^*_{A,\mathfrak{d}}$ are the harmonic functions associated to the collapsed process and to its adjoint. By (3.15),

$$\text{cap}(A,B) \;=\; \sup_H \Big\{ 2\langle L^* F_{A,B}\,,\, H\rangle_{\mu} \,-\, \langle H, (-L)H\rangle_{\mu} \Big\}\;,$$

where $F_{A,B}(x) = f_{A,\mathfrak{d}}(x)$, $x \in B^c$, $F_{A,B}(z) = 0$, $z \in B$. By construction of the collapsed process, $V_{A,\mathfrak{d}} = V_{A,B}$ and $V^*_{A,\mathfrak{d}} = V^*_{A,B}$ on B^c, where $V_{A,B}$ and $V^*_{A,B}$ are the harmonic functions of the original process. □

Proof of Lemma 2.6. Fix two disjoint subsets A, B of E and assume that B^c is finite. By Theorem 2.4, the capacity $\text{cap}(A,B)$ is given by (3.12). By the sector condition, the expression inside braces in this formula is bounded by

$$2\sqrt{C_0}\langle (-S)F\,,\, F\rangle_\mu^{1/2} \langle (-S)H\,,\, H\rangle_\mu^{1/2} \,-\, \langle H, (-S)H\rangle_\mu\;.$$

The supremum over H is thus bounded by $C_0\langle (-S)F\,,\, F\rangle_\mu$. Therefore,

$$\text{cap}(A,B) \;\le\; C_0 \inf_F \langle (-S)F\,,\, F\rangle_\mu\;,$$

where the infimum is carried over all functions F equal to 1 at A and 0 at B. By definition of the capacity in the reversible case, the right hand side is equal to $C_0\text{cap}^s(A,B)$. This proves the lemma in the case where the set B^c is finite. To extend it to the general case, it remains to apply Lemma 2.3. □

4. Flows and Proof of Theorem 2.7

We assume in this section that the state space E is finite. We first prove Theorem 2.7 in the case where the sets A and B are singletons. The proof relies on an identity, established in Lemma 4.1 below, which provides a variational formula for the norm $\langle f\,,\,\{[(-L)^{-1}]^s\}^{-1} f\rangle_\mu^{1/2}$.

Before stating this result, we start with an elementary observation. We claim that

$$\text{two gradient flows } \Psi_f,\ \Psi_g \text{ are equal if and only if } f-g \text{ is constant}\,. \tag{4.1}$$

Indeed, if the gradient flows are equal, since $\Psi_f - \Psi_g = \Psi_{f-g}$, in view of (2.13), $\langle (-L)(f-g), (f-g)\rangle_\mu = 0$ which implies that $f-g$ is constant. The converse is obvious.

Recall that we denote by $\mathcal{C}$ the set of divergence free flows and by Φ_f the flow associated to a function $f : E\to\mathbb{R}$ introduced in (2.16).

Lemma 4.1. *For every function $f : E\to\mathbb{R}$,*

$$\langle f\,,\,\{[(-L)^{-1}]^s\}^{-1} f\rangle_\mu \;=\; \langle L^* f, (-S)^{-1} L^* f\rangle_\mu \;=\; \inf_{\varphi\in\mathcal{C}} \|\Phi_f - \varphi\|^2\,.$$

Proof. Fix a function $f : E\to\mathbb{R}$. Since Φ_f is a flow, by (2.15) and by (4.1) there is a function $W : E\to\mathbb{R}$, unique up to an additive constant, and a unique divergence free flow Δ_f such that

$$\Phi_f \;=\; \Psi_W \;+\; \Delta_f\,.$$

Computing the divergences of each flow we obtain that $L^*f = SW$ so that $W = S^{-1}L^*f + c_0$ for some constant c_0. Therefore, since $\Psi_W = \Psi_{W+c}$ for any constant c, Ψ_V, with $V = S^{-1}L^*f$, is the the projection of the flow Φ_f on the space of gradient flows. Moreover, by (2.13),

$$\langle \Psi_V\,,\,\Psi_V\rangle \;=\; \langle V, (-S)V\rangle_\mu \;=\; \langle L^* f, (-S)^{-1} L^* f\rangle_\mu\,, \tag{4.2}$$

because $\langle L^*f, 1\rangle_\mu = 0$ as μ is invariant. Furthermore, since Ψ_V is the projection of the flow Φ_f on the space of gradient flows,

$$\langle \Psi_V\,,\,\Psi_V\rangle \;=\; \inf_{\varphi\in\mathcal{C}} \langle \Phi_f - \varphi\,,\,\Phi_f-\varphi\rangle\,,$$

which concludes the proof of the lemma. □

Lemma 4.2. *Fix a pair of points $a\neq b$ in E. Then,*

$$\mathrm{cap}(\{a\},\{b\}) \;=\; \inf_f\,\inf_{\varphi\in\mathcal{C}} \|\Phi_f-\varphi\|^2\,,$$

where the infimum is carried over all functions $f : E\to\mathbb{R}$ such that $f(a)=1$, $f(b)=0$. Moreover, the infimum is uniquely attained at

$$f = (1/2)\{V_{a,b} + V^*_{a,b}\}\,,\quad \varphi \;=\; (1/2)\left\{\Phi_{V^*_{a,b}} - \Phi^*_{V_{a,b}}\right\}\,, \tag{4.3}$$

provided for a function $g : E\to\mathbb{R}$ we denote by Φ^*_g the flow given by $\Phi^*_g(x,y) = g(x)c^*(x,y) - g(y)c^*(y,x)$.

Proof. The first assertion of the lemma follows from Lemmas 3.1 and 4.1. Moreover, the function f which solves the variational problem for the capacity coincides with the one which solves the variational problem (3.1). Hence, by Lemma 3.1, $(1/2)\{V_{a,b}+V^*_{a,b}\}$ is the unique functions which attains the minimum. It remains to show that $(1/2)\left\{\Phi_{V^*_{a,b}} - \Phi^*_{V_{a,b}}\right\}$ is the optimal divergence free flow.

Let $F = (1/2)\{V_{a,b} + V^*_{a,b}\}$. We claim that $(L^*F)(x) = (SV_{a,b})(x)$ for all $x \in E$. For $x \neq a$, b, this identity is obvious and has been derived in the proof of Lemma 3.1. For $x = a$, it reduces to the identity $\mathbb{P}^*_a[T^+_b < T^+_a] = \mathbb{P}_a[T^+_b < T^+_a]$ which, in view of (2.3), is equivalent to $\mathrm{cap}(\{a\},\{b\}) = \mathrm{cap}^*(\{a\},\{b\})$. Since this identity is the content of Lemma 2.3, and since the same argument applies to $x = b$, the claim is in force. In particular, by the proof of Lemma 4.1, $\Psi_{V_{a,b}}$ is the projection of the flow Φ_F on the space of gradient flows, and there is a unique divergence free flow Δ_F such that

$$\Phi_F \;=\; \Psi_{V_{a,b}} \;+\; \Delta_F \;, \quad \langle \Phi_F - \Delta_F \,,\, \Phi_F - \Delta_F \rangle \;=\; \inf_{\varphi \in \mathcal{C}} \langle \Phi_F - \varphi \,,\, \Phi_F - \varphi \rangle \;.$$

An elementary computations shows that $\Delta_F = \Phi_F - \Psi_{V_{a,b}} = (1/2)\{\Phi_{V^*_{a,b}} - \Phi^*_{V_{a,b}}\}$, which completes the proof of the lemma. □

We may restate the previous lemma to obtain a variational formula for the capacity in terms of the Dirichlet form.

Lemma 4.3. *Fix a pair of points $a \neq b$ in E. Then,*

$$\mathrm{cap}(\{a\},\{b\}) \;=\; \inf_V D(V) \;,$$

where the infimum is carried over all functions $V : E \to \mathbb{R}$ such that Ψ_V is the orthogonal projection on the space of gradient flows of some flow Φ_f with $f(a) = 1$, $f(b) = 0$. Moreover, the infimum is uniquely attained, up to additive constants, at $V = V_{a,b}$.

Proof. By Lemma 3.1,

$$\mathrm{cap}(\{a\},\{b\}) \;=\; \inf_f \langle L^* f, (-S)^{-1} L^* f\rangle_\mu \;,$$

where the infimum is carried over all functions $f : E \to \mathbb{R}$ such that $f(a) = 1$, $f(b) = 0$. To conclude the proof of the first assertion of the lemma it remains to recall identity (4.2).

To prove uniqueness of $V_{a,b}$, recall from the proof of Lemma 4.2 that $SV_{a,b} = L^*F$, where $F = (1/2)\{V_{a,b} + V^*_{a,b}\}$. Hence, by the proof of Lemma 4.1, $\Psi_{V_{a,b}}$ is the orthogonal projection of Φ_F. Therefore, $D(V_{a,b}) \geq \inf_V D(V)$. On the other hand, by (4.2) and since by Lemma 3.1, F is the optimal function, $D(V_{a,b}) \leq \inf_V D(V)$. This shows that $V_{a,b}$ is optimal.

To prove uniqueness, suppose that W is another optimal function, and that Ψ_W is the orthogonal projection on the space of gradient flows of some flow Φ_g with $g(a) = 1$, $g(b) = 0$. By the optimality of W and by (4.2)

$$\mathrm{cap}(\{a\},\{b\}) \;=\; D(W) \;=\; \langle L^* g, (-S)^{-1} L^* g\rangle_\mu \;.$$

Hence, by the uniqueness of Lemma 3.1, $g = F$, and by the proof of Lemma 4.1, $L^*F = SW$. Since L^*F is also equal to $SV_{a,b}$, we obtain that $SV_{a,b} = SW$, which implies that $V_{a,b} - W$ is constant, as claimed. □

The flows $\Phi_{V^*_{a,b}}$ and $\Phi^*_{V_{a,b}}$ which appear in the previous lemma have a simple probabilistic interpretation. Denote by $\{\mathbb{X}_n : n \geq 0\}$ the discrete time skeleton of the chain, and recall that $M(x) = \mu(x)\lambda(x)$ is a stationary state for $\mathbb{X}_n$, unique up to a multiplicative constant. For $B \subset E$, let G_B be the Green function of the process killed at B:

$$G_B(x,y) \;:=\; \mathbb{E}_x\Big[\sum_{n=0}^{\tau_B - 1} \mathbf{1}\{\mathbb{X}_n = y\}\Big]\;,$$

where τ_B (resp. τ_B^+) stands for the hitting time of (resp. return time to) B for the discrete time chain $\mathbb{X}_n$:

$$\tau_B \;=\; \min\{n \ge 0 : \mathbb{X}_n \in B\}\;, \quad \tau_B^+ \;=\; \min\{n \ge 1 : \mathbb{X}_n \in B\}\;.$$

In the same way, G^*_B, $B \subset E$, stands for the Green function of the time reversed chain killed at B.

Denote by $\mathbb{P}_x$, $x \in E$, the probability on path space $D(\mathbb{Z}_+, E)$ induced by the Markov chain $\{\mathbb{X}_n : n \ge 0\}$ starting from x, and by θ_n, $n \ge 0$, the time shift by n units of time. Fix two disjoint subsets A, B of E. By the last exit decomposition, for every $x \in E$,

$$\begin{aligned}
\mathbb{P}_x\big[\tau_A < \tau_B\big] \;&=\; \sum_{n\ge 0} \mathbb{P}_x\big[\mathbb{X}_n \in A\,,\, n < \tau_B\,,\, \tau_B^+ \circ \theta_n < \tau_A^+ \circ \theta_n\big] \\
&=\; \sum_{a\in A}\sum_{n\ge 0} \mathbb{P}_x\big[\mathbb{X}_n = a\,,\, n < \tau_B\big]\,\mathbb{P}_a\big[\tau_B^+ < \tau_A^+\big] \\
&=\; \sum_{a\in A} G_B(x,a)\,\mathbb{P}_a\big[\tau_B^+ < \tau_A^+\big]\;.
\end{aligned}$$

Since $M(x)G_B(x,y) = M(y)G^*_B(y,x)$, it follows from the previous identity

$$V_{A,B}(x) \;=\; \mathbb{P}_x\big[\tau_A < \tau_B\big] \;=\; \sum_{a\in A} \frac{1}{M(x)}\, G^*_B(a,x)\, M(a)\, \mathbb{P}_a\big[\tau_B^+ < \tau_A^+\big]\;. \tag{4.4}$$

Denote by $\nu_{A,B}$ the harmonic measure, also called the normalized charge distribution,

$$\nu_{A,B}(a) \;=\; \frac{1}{\mathrm{cap}(A,B)}\, M(a)\, \mathbb{P}_a\big[\tau_B^+ < \tau_A^+\big]\;.$$

Fix two disjoint subsets A, B of E. Denote by $i(x,y) = i_{A,B}(x,y)$ the current through the arc (x,y) for the process which starts from the harmonic measure $\nu_{A,B}$ and which is killed at B:

$$i(x,y) \;:=\; \mathbb{E}_{\nu_{A,B}}\Big[\sum_{n=0}^{\tau_B-1} \big\{\mathbf{1}\{\mathbb{X}_n = x, \mathbb{X}_{n+1} = y\} - \mathbf{1}\{\mathbb{X}_n = y, \mathbb{X}_{n+1} = x\}\big\}\Big]\;.$$

By the Markov property and in view of (4.4), if we denote by $i^*(x,y)$ the current through the arc (x,y) for the time reversed chain,

$$\begin{aligned}
i^*(x,y) \;&:=\; \sum_{a\in A} \nu_{A,B}(a)\{G^*_B(a,x)p^*(x,y) \;-\; G^*_B(a,y)p^*(y,x)\} \\
&=\; \sum_{a\in A} \nu_{A,B}(a)\Big\{\frac{1}{M(x)} G^*_B(a,x)c^*(x,y) \;-\; \frac{1}{M(y)} G^*_B(a,y)c^*(y,x)\Big\} \qquad (4.5) \\
&=\; \mathrm{cap}(A,B)^{-1}\big\{V_{A,B}(x)\, c^*(x,y) \;-\; V_{A,B}(y)\, c^*(y,x)\big\}\;.
\end{aligned}$$

Since this last expression is equal to $\mathrm{cap}(A,B)^{-1}\Phi^*_{V_{A,B}}(x,y)$, $\Phi^*_{V_{A,B}}$ is, up to the multiplicative constant $\mathrm{cap}(A,B)$, the current through the arc (x,y) for the time reversed Markov chain $\mathbb{X}^*_n$ started from the harmonic measure $\nu_{A,B}$ and killed at B.

Analogously, $\Phi_{V^*_{A,B}}$ is, up to the same multiplicative constant, the current through the arc (x,y) of the discrete time Markov chain $\mathbb{X}_n$ started from the harmonic measure $\nu^*_{A,B}$ and killed at B.

Given a function $f : E \to \mathbb{R}$, we may write the flow Φ_f as $\Phi_f = \Psi_f + \Upsilon_f$, where Υ_f is the flow given by

$$\Upsilon_f(x,y) \;=\; c_a(x,y)\,\{f(x) + f(y)\}\ .$$

It turns out that the flows Ψ_f and Υ_f are orthogonal:

$$\langle \Psi_f, \Upsilon_f\rangle \;=\; \frac{1}{2} \sum_{(x,y)\in\mathcal{E}} \frac{1}{c_s(x,y)}\, \Psi_f(x,y)\, \Upsilon_f(x,y) \;=\; 0\ . \tag{4.6}$$

Indeed, by definition of the flows Ψ_f and Υ_f,

$$\sum_{(x,y)\in\mathcal{E}} \frac{1}{c_s(x,y)}\, \Psi_f(x,y)\, \Upsilon_f(x,y) \;=\; \sum_{x,y\in E} c_a(x,y)\,\{f(x)^2 - f(y)^2\}\ .$$

Since $c_a(x,y) = (1/2)\{c(x,y) - c(y,x)\} = (1/2)\{c(x,y) - c^*(x,y)\}$, the previous expression is equal to

$$\frac{1}{2} \sum_{x\in E} M(x)\,(I-P)f^2(x) \;-\; \frac{1}{2} \sum_{x\in E} M(x)(I-P^*)f^2(x) \;=\; 0\ ,$$

where P represents the operator in $L^2(M)$ defined by $(Pg)(x) = \sum_{y\in E} p(x,y)g(y)$, and where P^* stands for the adjoint of P in $L^2(M)$. This proves (4.2).

This orthogonality permits to restate Lemma 4.2 in a slightly different form, quite useful in some cases.

Lemma 4.4. *Fix a pair of points $a \neq b$ in E. Then,*

$$\mathrm{cap}(\{a\},\{b\}) \;=\; \inf_f \inf_{\varphi\in\mathcal{C}} \big\{ D(f) \;+\; \|\Upsilon_f - \varphi\|^2 \big\}\ ,$$

where the infimum is carried over all functions $f : E \to \mathbb{R}$ such that $f(a) = 1$, $f(b) = 0$.

We are now ready to prove Theorem 2.7.

Proof of Theorem 2.7. We proceed in two steps, collapsing each set at a time. Fix two disjoint subsets A, B of E and recall the notation introduced around (3.9). We first prove that

$$\inf_F \inf_\varphi \|\Phi_F - \varphi\|^2 \;=\; \inf_f \inf_\psi \|\Phi_f - \psi\|_A^2\ , \tag{4.7}$$

where the infimum on the left hand side is carried over all functions $F : E \to \mathbb{R}$ constant over A and flows φ such that $(\mathrm{div}\ \varphi)(x) = 0$, $x \in A^c$, $\sum_{x\in A}(\mathrm{div}\ \varphi)(x) = 0$; while on the right hand side $\|\cdot\|_A$ represents the norm associated to the scalar product introduced in (2.12) on the set $\overline{E}_A$ for the process $\overline{X}_t$ and the infimum is carried over all functions $f : \overline{E}_A \to \mathbb{R}$ and divergence free flows ψ on $\overline{E}_A$.

Consider a function $F : E \to \mathbb{R}$ constant in A and a flow φ on E such that $(\mathrm{div}\ \varphi)(x) = 0$, $x \in A^c$, $\sum_{x\in A}(\mathrm{div}\ \varphi)(x) = 0$. Recall the definition of the function $f : \overline{E}_A \to \mathbb{R}$ introduced below (3.8) and let ψ be the flow on $\overline{E}_A$ given by

$$\psi(x,y) \;=\; \varphi(x,y)\ , \quad \psi(x,\mathfrak{d}) \;=\; \sum_{y\in A} \varphi(x,y)\ , \quad x\,,y\in A^c\ .$$

One checks that ψ is a divergence free flow. Moreover, by Schwarz inequality,

$$\|\Phi_f - \psi\|_A^2 \;\le\; \|\Phi_F - \varphi\|^2 \;.$$

It follows from this estimate that the left hand side of (4.7) is greater than or equal to the right hand side.

Conversely, fix a function $f : \overline{E}_A \to \mathbb{R}$ and a divergence free flow ψ on $\overline{E}_A$. Let $F : E \to \mathbb{R}$ be the function defined above (3.8), and let φ be the flow in E given by

$$\varphi(x,y) \;=\; \psi(x,y)\;, \quad x\,,y\in A^c\;, \qquad \varphi(z,w) \;=\; 2f(\mathfrak{d})\,c_a(z,w)\;, \quad z\,,w\in A\;,$$
$$\varphi(x,y) \;=\; \Phi_F(x,y) \;-\; \frac{c_s(x,y)}{\sum_{z\in A} c_s(x,z)}\Big\{\sum_{z\in A}\Phi_F(x,z) - \psi(x,\mathfrak{d})\Big\}\;, \quad x\in A^c\,, y\in A\;.$$

One checks that $(\mathrm{div}\ \varphi)(x) = 0$, $x\in A^c$, that $\sum_{x\in A}(\mathrm{div}\ \varphi)(x) = 0$, and that $\|\Phi_F - \varphi\| = \|\Phi_f - \psi\|_A$. Therefore, the left hand side of (4.7) is less than or equal to the right hand side, proving claim (4.7).

We are now in a position to prove the theorem. Fix a site $x\in E$ and a set $A\not\ni x$. By (3.10), $\mathrm{cap}(\{x\},A) = \overline{\mathrm{cap}}(\{x\},\{\mathfrak{d}\})$. The assertion of the theorem when the set B is a singleton follows from Lemma 4.2 and (4.7). The general case is proved analogously by first collapsing the set A and then collapsing the set B. □

We conclude this section with a bound on the capacity in the denumerable case. Assume that E is a countable set, fix a site $x\in E$ and a set $B\not\ni x$, with B^c finite. Then,

$$\mathrm{cap}(\{x\},B) \;\le \tag{4.8}$$
$$\inf_F\Big\{D(F) \;+\; \frac{1}{2}\sum_{\substack{(x,y)\in\mathcal{E}\\ x,y\in B^c}}\frac{c_a(x,y)^2}{c_s(x,y)}\{\,F(x)+F(y)-2\,\}^2 \;+\; 4\sum_{\substack{(x,z)\in\mathcal{E}\\ x\in B^c, z\in B}}\frac{c_a(x,z)^2}{c_s(x,z)}\Big\},$$

where the infimum is carried over all functions $F : E\to\mathbb{R}$ such that $F(x) = 1$, $F(z) = 0$, $z\in B$.

Indeed, recall the notation introduced around (3.13). Denote by $\overline{E}$ the set $B^c\cup\{\mathfrak{d}\}$, where $\mathfrak{d}$ is an extra site added to E. Let $\{\overline{X}_t : t\ge 0\}$ be the process obtained from X_t by collapsing the set B to the point $\mathfrak{d}$, and let $\overline{D}$, $\overline{\mathcal{E}}$, $\overline{c}(x,y)$ and $\overline{\mathrm{cap}}$ be the associated Dirichlet form, oriented bonds, conductances and capacities, respectively.

By (3.14), $\mathrm{cap}(\{x\},B) = \overline{\mathrm{cap}}(\{x\},\{\mathfrak{d}\})$. Fix a function $F : E\to\mathbb{R}$ which vanishes on B and is equal to 1 at x, and let $f : \overline{E}\to\mathbb{R}$ be given by $f(y) = F(y)$, $y\in B^c$, $f(\mathfrak{d}) = 0$. Since $\overline{c}_a(x,y)$ is a divergence free flow, by Lemma 4.4,

$$\overline{\mathrm{cap}}(0,\mathfrak{d}) \;\le\; \overline{D}(f) \;+\; \frac{1}{2}\sum_{(x,y)\in\overline{\mathcal{E}}}\frac{1}{\overline{c}_s(x,y)}\big\{\overline{c}_a(x,y)[f(x)+f(y)] - 2\overline{c}_a(x,y)\big\}^2\;.$$

Clearly, $\overline{D}(f) = D(F)$. On the other hand, by (3.13) if the arc (x,y) is contained in B^c, we may replace $\overline{c}_s(x,y)$, $\overline{c}_a(x,y)$ and f by $c_s(x,y)$, $c_a(x,y)$ and F, respectively. In contrast, for an arc $(x,\mathfrak{d})$, $x\in B^c$, since $\overline{c}_t(x,\mathfrak{d}) = \sum_{z\in B} c_t(x,z)$, $t = a,s$, and since $f(\mathfrak{d}) = 0 = F(z)$, $z\in B$, by Schwarz inequality,

$$\Big\{\overline{c}_a(x,\mathfrak{d})\big[\,f(x)+f(\mathfrak{d})-2\,\big]\Big\}^2 \;=\; \Big\{\sum_{z\in B} c_a(x,z)\big[\,F(x)+F(z)-2\,\big]\Big\}^2$$
$$\le\; \sum_{z\in B}\frac{c_a(x,z)^2}{c_s(x,z)}[F(x)+F(z)-2]^2\sum_{z\in B} c_s(x,z)\;.$$

Since $0 \le F \le 1$, $F(x) + F(z) - 2$ is absolutely bounded by 2. Putting together all previous estimates we derive (4.8).

5. Recurrence criteria

It is well known that in the reversible case the Dirichlet and the Thomson principle provide powerful tools to prove the recurrence or the transience of irreducible Markov processes evolving in countable state spaces. In this section, we examine this matter in the non reversible case.

Consider a irreducible Markov process $\{X_t : t \ge 0\}$ on a countable state space E satisfying the assumptions of the beginning of Section 2. We assume, in particular, the existence of a stationary state μ.

It is well known that the Markov process X_t is recurrent if and only if there exist a site $0 \in E$ and a sequence of *finite* subsets B_n containing 0 and increasing to E, $B_n \subset B_{n+1}$, $\cup_n B_n = E$, such that

$$\lim_{n\to\infty} \mathbb{P}_0\big[T_{B_n^c} < T_0^+\big] \;=\; 0 \;.$$

By (2.3), for any finite set B containing the site 0,

$$\frac{1}{M(0)}\, \mathbb{P}_0\big[T_{B^c} < T_0^+\big] \;=\; \mathrm{cap}(0, B^c) \;.$$

Hence, the Markov process X_t is recurrent if and only if there exist a site $0 \in E$ and a sequence of finite subsets B_n containing 0 and increasing to E such that

$$\lim_{n\to\infty} \mathrm{cap}(0, B_n^c) \;=\; 0 \;. \tag{5.1}$$

The proof of the recurrence is thus reduced to the estimation of the capacity between a site and the complement of a finite set. This problem can be further simplified by collapsing the set B_n^c to a point, as we did in Section 3.

The first two results follow from the previous observation and the bounds stated in Lemmas 2.5 and 2.6. Recall that $\{X_t^s \,|\, t \ge 0\}$ stands for the reversible version of the process X_t whose generator is given by S.

Lemma 5.1. *Let $\{X_t \,|\, t \ge 0\}$ be a irreducible Markov process on a countable state space E which admits a stationary measure. The process is transient if so is the Markov process $\{X_t^s \,|\, t \ge 0\}$.*

Lemma 5.2. *Let $\{X_t \,|\, t \ge 0\}$ be a irreducible Markov process on a countable state space E which admits a stationary measure. The process is recurrent if its generator satisfies a sector condition and if the Markov process $\{X_t^s \,|\, t \ge 0\}$ is recurrent.*

Cycle random walks with bounded cycles, [13], [12], mean zero asymmetric exclusion process [16], or asymmetric zero range process on a finite cylinder [9] are examples of non reversible Markov processes which satisfy the sector condition.

Lemma 5.3. *Let $\{X_t \,|\, t \ge 0\}$ be a irreducible Markov process on a countable state space E which admits a stationary measure. The process is recurrent if the Markov process $\{X_t^s \,|\, t \ge 0\}$ is recurrent and if*

$$\sum_{(x,y)\in\mathcal{E}} \frac{c_a(x,y)^2}{c_s(x,y)} \;<\; \infty \;.$$

Proof. Fix $\epsilon > 0$ and a site $0 \in E$. By assumption, there exists a finite set $A \ni 0$ such that

$$\sum_{\substack{(x,y)\in\mathcal{E}\\ \{x,y\}\not\subset A}} \frac{c_a(x,y)^2}{c_s(x,y)} \;\le\; \epsilon\;.$$

By (2.3), for all subsets B of E such that $A \subset B^c$, B^c finite, $\text{cap}^s(A,B) \le \sum_{x\in A} \text{cap}^s(\{x\},B)$. Hence, since the process X^s_t is recurrent, by (5.1) and by Lemma 2.2, there exists a finite set $B^c \supset A$ such that $\text{cap}^s(A,B) \le \epsilon$.

Denote by $V^s_{A,B} : E \to \mathbb{R}$ the equilibrium potential associated to the reversible process X^s: $V^s_{A,B}(x) = \mathbb{P}^s_x[T_A < T_B]$. Since $D(V^s_{A,B}) = \text{cap}^s(A,B)$, by construction of B, $D(V^s_{A,B}) \le \epsilon$. Therefore, by (4.8) with $F = V^s_{A,B}$, $\text{cap}(0,B)$ is bounded above by

$$\epsilon \;+\; \frac{1}{2} \sum_{\substack{(x,y)\in\mathcal{E}\\ x,y\in B^c}} \frac{c_a(x,y)^2}{c_s(x,y)} \big\{\, V^s_{A,B}(x) + V^s_{A,B}(y) - 2\,\big\}^2 \;+\; 4 \sum_{\substack{(x,z)\in\mathcal{E}\\ x\in B^c, z\in B}} \frac{c_a(x,z)^2}{c_s(x,z)}\;\cdot$$

Since $V^s_{A,B}$ is identically equal to 1 on A, the previous expression is less than or equal to

$$\epsilon \;+\; 4 \sum_{\substack{(x,y)\in\mathcal{E}\\ x\in A, y\in A^c}} \frac{c_a(x,y)^2}{c_s(x,y)} \;+\; 2 \sum_{\substack{(x,y)\in\mathcal{E}\\ x,y\in A^c}} \frac{c_a(x,y)^2}{c_s(x,z)}\;\cdot$$

By definition of the set A, this expression is bounded by 5ϵ, which concludes the proof of the lemma. □

5.1. **Random walks with self-similar rescaled invariant potential.** In [8], Durrett built from a random potential, with a large scale self-similarity property, a reversible nearest-neighbor random walk on $\mathbb{Z}^d$ for which Sinai random walk is a special case when $d = 1$. He proved that such a random walk is recurrent under simple and natural assumptions on the scaling limit of the potential. Note, however, that such a random walk could never be a particular case of classical randoms walks in random environment in dimension $d \ge 2$ due to the reversibility condition.

We want to point out here that the key feature of this model is *not* the reversibility but, as Durrett suggested, the existence of a strongly fluctuating invariant measure. The reason for the restriction to the reversible was that it allowed the use of the Dirichlet principle. Our extended Dirichlet principle permits to reproduce Durrett's argument with only assumptions on the invariant measure and without the reversibility hypothesis.

Consider a discrete time, nearest-neighbor random walk $\{X_n \,|\, n \ge 1\}$ on $\mathbb{Z}^d$ with random transition probabilities $p(x,y) : \Omega \to [0,1]$, and assume the existence of a (random) invariant measure μ. We define the invariant potential $V : \mathbb{Z}^d \to \mathbb{R}$ by

$$\mu(x) \;=\; e^{-V(x)} \qquad x \in \mathbb{Z}^d\;,$$

and assume, without loss of generality, that $V(0) = 0$. The relation between the random potential and the invariant measure in [8] is not exactly the same, but this definition makes our point more clear. We can extend V into a continuous function $V : \mathbb{R}^d \to \mathbb{R}$ and see it as a random variable in $C(\mathbb{R}^d, \mathbb{R})$, the space of continuous functions from $\mathbb{R}^d$ to $\mathbb{R}$ equipped with the topology of uniform convergence on compact sets. We assume that there exists $\alpha > 0$ and a random variable $W : \Omega \to$

$C(\mathbb{R}^d, \mathbb{R})$ such that $\lambda^{-\alpha} V(\lambda\,\cdot\,)$ converges in law to $W(\,\cdot\,)$ when $\lambda \uparrow \infty$. Hence, W is a self-similar d-dimensional process and we have the following result.

Lemma 5.4. *If there is almost surely $a > 0$ such that the connected component of the origin in $\{x \in \mathbb{R}^d : W(x) < a\}$ is bounded, then X is almost surely recurrent.*

We refer to [8] for examples of (reversible) processes which satisfy such hypotheses. Even though we could relax Durrett's reversibility hypothesis, it is not clear how to build non artificial irreversible examples in which one has enough control on an invariant measure to check the assumptions on V. One can start, for example, as in [8], with a random potential V with stationary increments, and build the reversible random walk inside this potential to have μ as invariant measure. We may then add some irreversibility by superposing to this reversible dynamics some drift along cycles on the level sets of V, keeping μ as an invariant measure.

Proof of Lemma 5.4. We follow closely Durrett's proof. First, following Skorohod [14], we can build on the same probability space random variables $V_1, V_2, \dots$ with the same law as V and such that $n^{-\alpha} V_n(n\,\cdot\,)$ converges *almost surely* in $C(\mathbb{R}^d, \mathbb{R})$ to $W(\,\cdot\,)$. Define C_a, $a > 0$, as the bounded connected component of the origin in $\{x \in \mathbb{R}^d : W(x) < a\}$, and set

$$G_n = (nC_a) \cap \mathbb{Z}^d \ , \quad \partial_- G_n = \{x \in G_n : \exists y \notin G_n \ , \|x - y\| = 1\} \ ,$$

where $\|\cdot\|$ stands for the Euclidean norm.

We claim that $\mu_n(\partial_- G_n)$ converges almost surely to 0, where $\mu_n(x) = e^{-V_n(x)}$, $x \in \mathbb{Z}^d$. Indeed, by assumption there are r, $R > 0$ such that

$$C_a \subset [-R, R]^d \ , \quad W(y) \ge a/2$$

for all $y \in B(z, r)$, $z \in \partial C_a$, where $B(x, r)$ stands for the ball centered at x with radius r and ∂C_a for the boundary of C_a. Therefore, almost surely, for n large enough,

$$\mu_n(\partial_- G_n) \;=\; \sum_{x \in \partial_- G_n} e^{-V_n(x)} \;\le\; |G_n|\, e^{-(a/4)n^\alpha} \;\le\; n^d (2R)^d e^{-(a/4)n^\alpha} \ ,$$

which proves the claim.

For any finite subset B of $\mathbb{Z}^d$ which contains the origin, since $\mu(0) = e^{-V(0)} = 1$, we have

$$\mathbb{P}_0[\tau_0^+ = \infty] \;\le\; \mu(0)\mathbb{P}_0[\tau_0^+ > \tau_{B^c}^+] \;=\; \mathrm{cap}(0, B^c) \ .$$

By taking the test function $\mathbf{1}\{B\}$ in (4.8) we obtain that $\mathrm{cap}(0, B^c)$ is bounded above by

$$\sum_{x \in B, y \in B^c} c_s(x, y) \;+\; 4 \sum_{x \in B, y \in B^c} \frac{c_a(x, y)^2}{c_s(x, y)} \;\le\; 5 \sum_{x \in B, y \in B^c} c_s(x, y) \;\le\; 5\mu(\partial_- B) \ .$$

Since $\mu(0) = 1$ we also have $\mathrm{cap}(0, B^c) \le 1$, thus $\mathrm{cap}(0, B^c) \le 5\mu(\partial_- B) \wedge 1$. Now, for any $k > 0$,

$$\begin{aligned} P_0(\tau_0^+ = +\infty) \quad &\le \quad \min\left\{\mathrm{cap}(0, B^c) \ge 0 : \ 0 \in B \subset [-k, k]^d\right\} \\ &\le \quad \min_{0 \in B \subset [-k,k]^d} 5\mu(\partial_- B) \wedge 1. \end{aligned}$$

Since V_n has the same distribution as V, taking expected values with respect to the environment, denoted by $\mathbf{E}$, by the monotone convergence theorem, we obtain that, for all $n \geq 1$,

$$\begin{aligned}\mathbf{E}\big[\,\mathbb{P}_0[\tau_0^+ = +\infty]\,\big] \;&\leq\; \lim_{k\to\infty} \mathbf{E}\Big[\min_{0\in B\subset[-k,k]^d} 5\mu(\partial_- B)\wedge 1\Big] \\ &= \lim_{k\to\infty} \mathbf{E}\Big[\min_{0\in B\subset[-k,k]^d} 5\mu_n(\partial_- B)\wedge 1\Big] \\ &= \mathbf{E}\Big[\lim_{k\to\infty}\min_{0\in B\subset[-k,k]^d} 5\mu_n(\partial_- B)\wedge 1\Big] \;\leq\; \mathbf{E}\Big[5\mu_n(\partial_- G_n)\wedge 1\Big]\;.\end{aligned}$$

Thus, by the dominated convergence theorem,

$$\begin{aligned}\mathbf{E}\big[\,\mathbb{P}_0[\tau_0^+ = +\infty]\,\big] \;&\leq\; \lim_{n\to+\infty} \mathbf{E}\big[\,5\mu_n(\partial_- G_n)\wedge 1\,\big] \\ &=\; \mathbf{E}\Big[\lim_{n\to+\infty} 5\mu_n(\partial_- G_n)\wedge 1\Big] \;=\; 0\;,\end{aligned}$$

which proves that the process is almost surely recurrent. □

5.2. **Two dimensional random walk in asymmetric random conductances.** The most natural way to generalize the classical random conductance model on a graph may be the following. To define the asymmetric conductances $c(x,y)$ on each arc (x,y) we superpose symmetric functions $c_s(x,y)$ and a divergence free flow $c_a(x,y)$ with the restriction that $|c_a| \leq c_s$ to end with nonnegative conductances $c(x,y)$.

More precisely, consider a family Γ of finite cycles γ on a countable graph $(E,\mathcal{E})$, and a family of nonnegative random variables Z_γ, $\gamma\in\Gamma$, such that for each $(x,y)\in\mathcal{E}$,

$$\sum_{\gamma\in\Gamma} Z_\gamma\, |\chi_\gamma(x,y)| \;<\; \infty\;,$$

where χ_γ is the divergence free flow introduced in (2.14). Define the divergence free flow c_a by

$$c_a(x,y) \;=\; \sum_{\gamma\in\Gamma} Z_\gamma\, \chi_\gamma(x,y)\;, \quad (x,y)\in\mathcal{E}\;.$$

There are two natural ways to define symmetric conductances in this context. Consider a family of nonnegative random variables $\{Y_{(x,y)} : (x,y)\in\mathcal{E}\}$ such that $Y_{(x,y)} = Y_{(y,x)}$. We may set $c_s(x,y) = Y_{(x,y)} + |c_a(x,y)|$, or $c_s(x,y) = Y_{(x,y)} + \sum_{\gamma\in\Gamma} Z_\gamma\, |\chi_\gamma(x,y)|$.

In the special case of the two dimensional lattice, we can decompose each flow associated to a finite cycle as a linear combination of elementary flows associated to cycles of length 4. For $x\in\mathbb{Z}^2$, denote by γ_x the cycle $(x, x+e_1, x+e_1+e_2, x+e_2, x)$, where e_1, e_2 stands for the canonical basis of $\mathbb{R}^2$. A flow χ_γ associated to a finite cycle γ can be written as

$$\chi_\gamma \;=\; \sum_{x\in\mathbb{Z}^d} W_{\gamma,\gamma_x}\, \chi_{\gamma_x}\;,$$

where $W_{\gamma,\gamma_x} = 1$ (resp. -1) if the cycle γ_x is contained in the interior of γ and the cycle γ runs counter-clockwise (resp. clockwise), and $W_{\gamma,\gamma_x} = 0$ if the cycle γ_x is not contained in the interior of γ.

Denote by $\mathbf{E}$ expectation with respect to the random variables Z_γ and assume that

$$\sum_{\gamma\in\Gamma} \mathbf{E}[Z_\gamma]\, |W_{\gamma,\gamma_x}| \;<\; \infty \quad \text{for all } x\in\mathbb{Z}^d\;.$$

In this case $W_{\gamma_x} := \sum_{\gamma\in\Gamma} Z_\gamma\, W_{\gamma,\gamma_x}$ is almost surely well defined for all $x \in \mathbb{Z}^d$ and so is the divergence free flow c_a given by

$$c_a \;=\; \sum_{x\in\mathbb{Z}^d} W_{\gamma_x}\, \chi_{\gamma_x} \;. \tag{5.2}$$

Note that each arc (x, y) belongs to exactly two elementary cycles, denoted by $\gamma^{\pm}(x, y)$ and characterized by the fact that $\chi_{\gamma^{\pm}(x,y)}(x, y) = \pm 1$. With this notation, for any arc (x, y), $c_a(x, y) = W_{\gamma^+(x,y)} - W_{\gamma^-(x,y)}$.

Lemma 5.5. *Suppose that*

$$\sup_{(x,y)} \mathbf{E}\Big[\, c_s(x, y) \;+\; \frac{[W_{\gamma^+(x,y)}]^2 + [W_{\gamma^-(x,y)}]^2}{c_s(x, y)} \,\Big] \;<\; \infty \;,$$

where the supremum is carried over all arcs. Then, the random walk is almost surely recurrent.

Proof. Let $B_n^c = [-n, n]^2$, $n \ge 1$, consider a function $f_n : \mathbb{Z}^2 \to \mathbb{R}$ such that $f_n(0) = 1$, $f_n(x) = 0$ for $x \in B_n$, and a divergence free flow $\psi_n = \sum_x a_x \chi_{\gamma_x}$, where the sum is performed over all $x \in \mathbb{Z}^2$ for which the elementary cycle γ_x is contained in B_n^c. Repeating the proof of (4.8) and keeping in mind that c_a is absolutely bounded by c_s, we obtain that

$$\mathrm{cap}(0, B_n) \;\le\; D(f_n) \;+\; \frac{1}{2} \sum_{x,y\in B_n^c} \frac{1}{c_s(x, y)} \Big\{ c_a(x, y)[f_n(x) + f_n(y)] - \psi_n(x, y) \Big\}^2 \;.$$

Consider the divergence free flow φ_n given by

$$\varphi_n \;=\; \frac{1}{2} \sum_{x} F_n(\gamma_x)\, W_{\gamma_x}\, \chi_{\gamma_x} \;, \quad \text{where} \quad F_n(\gamma_x) \;=\; \sum_{z\in\gamma_x} f_n(z) \;,$$

and where the sum is carried over all sites x in $\mathbb{Z}^2$ for which the elementary cycle γ_x is contained in B_n^c. By the previous bound,

$$\mathrm{cap}(0, B_n) \;\le\; D(f_n) \;+\; \frac{1}{2} \sum_{x,y\in B_n^c} \frac{1}{c_s(x, y)} \Big\{ c_a(x, y)[f_n(x) + f_n(y)] - \varphi_n(x, y) \Big\}^2 \;.$$

As we know, $D(f_n) = (1/2) \sum_{x,y\in\mathbb{Z}^2} c_s(x, y)[f_n(y) - f_n(x)]^2$. On the other hand, it follows from the definitions of the asymmetric conductance and the divergence free flow φ_n that $c_a(x, y)[f_n(x) + f_n(y)] - \varphi_n(x, y)$ is equal to $W_{\gamma^+(x,y)}\{f_n(x) + f_n(y) - (1/2)F_n(\gamma^+(x, y))\} - W_{\gamma^-(x,y)}\{f_n(x) + f_n(y) - (1/2)F_n(\gamma^-(x, y))\}$ if the arc (x, y) does not belong to one side of the square B_n^c. The absolute value of this difference is bounded above by $|W_{\gamma^+(x,y)}| \max_{e\in\gamma^+(x,y)} |f_n(e^+) - f_n(e^-)| + |W_{\gamma^-(x,y)}| \max_{e\in\gamma^-(x,y)} |f_n(e^+) - f_n(e^-)|$. If the arc (x, y) belongs to one side of the square B_n^c, taking advantage of the fact that f_n vanishes outside B_n^c, we obtain a similar formula with an extra factor 2. In conclusion, $\mathrm{cap}(0, B_n)$ is bounded above by

$$4 \sum_{x,y\in\mathbb{Z}^2} \Big\{ c_s(x, y) + \frac{[W_{\gamma^-(x,y)}]^2 + [W_{\gamma^+(x,y)}]^2}{c_s(x, y)} \Big\} \max_e [f_n(e^+) - f_n(e^-)]^2 \;,$$

where the maximum is carried over all arcs e in $\gamma^-(x, y) \cup \gamma^+(x, y)$.

Let

$$f_n(x) \;=\; \Big(1 - \frac{\log(1 + \|x\|_\infty)}{\log(n + 2)}\Big) \mathbf{1}\{[-n, n]^2\}(x) \;,$$

where $\|x\|_\infty = \max\{|x_1|, |x_2|\}$, $x = (x_1, x_2)$. It follows from this choice and from the assumption of the lemma that

$$\lim_{n\to\infty} \mathbf{E}[\mathrm{cap}(0, B_n)] \;=\; 0\;.$$

In particular, there exists almost surely a subsequence $(B_{n_k} : k \ge 1)$ such that $\lim_{k\to\infty} \mathrm{cap}(0, B_{n_k}) = 0$ and the almost sure recurrence follows. □

We conclude with an example which satisfies the assumptions of the previous lemma. Suppose that the random variables Z_γ are independent Poisson variables with parameter $\lambda^{|\gamma|}$, $0 < \lambda < 1/3$, and that the random variables $Y_{(x,y)}$ have a common distribution bounded away from 0 and with a finite first moment. Let c_a be given by (5.2) and let $c_s(x, y) = Y_{(x,y)} + |c_a(x, y)|$. We claim that the hypotheses of the previous result are fulfilled.

Indeed, by assumption there exists $\delta > 0$ such that $c_s(x, y) \ge Y_{(x,y)} \ge \delta > 0$ almost surely. Therefore, to show that the assumptions of the previous lemma are in force we need only to prove that

$$\sup_{(x,y)} \mathbf{E}\big[c_s(x, y)\big] \;<\; \infty \quad\text{and}\quad \sup_{x\in\mathbb{Z}^d} \mathbf{E}\big[W^2_{\gamma_x}\big] \;<\; \infty\;. \tag{5.3}$$

Since $|c_a(x, y)| \le W_{\gamma^+(x,y)} + W_{\gamma^-(x,y)}$, for every arc (x, y),

$$\mathbf{E}\big[c_s(x, y)\big] \;\le\; \mathbf{E}\big[Y(x, y)\big] \;+\; \sum_{p=\pm}\sum_{\gamma\in\Gamma} W_{\gamma,\gamma^p(x,y)}\, \mathbf{E}\big[Z_\gamma\big]\;.$$

By assumption, the first term on the right hand side is bounded uniformly over (x, y), while the second term is less than or equal to $\sum_{k\ge4} 8k3^k\lambda^k$ because there are at most $4 \cdot 3^{k-1}$ self-avoiding walks of length k and because a cycle of length k containing in its interior an elementary cycle must cross a line parallel to one of the axis in at most $2k$ points. This proves the first bound in (5.3). To prove the second bound, fix an elementary cycle γ_x. By definition of the random variables Z_γ,

$$\mathbf{E}\big[W^2_{\gamma_x}\big] \;=\; \mathbf{E}\big[W_{\gamma_x}\big]^2 \;+\; \sum_{\gamma\in\Gamma} \lambda^{|\gamma|}\, W^2_{\gamma,\gamma_x}\;.$$

The first expectation has been estimated above, while the second one can be estimated in the same way. This concludes the proof of (5.3).

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